Musical Instruments and Twelve Tone Equal Temperament

C. Parks Poughkeepsie NY

→ Center for Lifetime Study, Marist CLS

Presented Sept. 3, 2024
Version for study later

Musical Instruments and Twelve Tone Equal Temperament

C. Parks Placeholder for email header comments

"Musical Instruments and Twelve Tone Equal Temperament" C. Parks Sept 2024 Poughkeepsie, NY
Presentation within Science Potpourri at Marist Center for Lifelong Studies. ChrisParksCars@gmail.com

Pythagoras created a genius musical notes system containing an inherent math-based flaw. I presented abridged slides (marked with blue stars) to a general audience. The full slides here give the more careful treatment for musicians (choir directors, conservatory teachers, or string students.) This presentation, as viewed from an outside math perspective, shows how musicians (Monteverdi, Bach, Casals) creatively work around an equally genius but also flawed equal-temperament solution.
I was delighted that musicologists uncovered in 2005 that Bach crafty-ducked TET in his famous 48 Klavier pieces!

As a child I was fascinated with our twelve-tone equal temperament musical scale, which forces all keys "equivalent." C versus C sharp or F sharp are transposed by sliding notes. But I also knew the musical notation is mysteriously super-redundant. Also, composers are passionate about particular keys, for instance Beethoven picking his quartet in C sharp minor. Attached is a math-oriented non-musician's take on our classical musical notes system!

6th century BC **Pythagoras** defined the octave as a factor of two frequency change (A440, A220, or A110 beats per second.) Pythagoras set the musical "fifth" as a "perfect" frequency ratio of 3/2. From these simple assumptions Pythagoras then astonishingly calculates the frequencies of all twelve notes of the scale in powers of two and 3/2. **The Pythagoras musical system, with rich resonances and overtones, is one of the ten greatest mathematical achievements of the ancient world!** But this genius system contains an inherent flaw: the octave 2.0 ratio is slightly (1.5% or a few nasty "beats") incompatible with the nice integer ratios! Resonances and overtones are perfect, but composers are crippled to only use particular keys.
I claim Pythagoras system is "simple genius" – after carefully viewing 2nd to last page of presentation may you also experience an exciting "I got it" moment

My wife and I heard the spectacular Stockholm reconstructed 1651 *Düben* Organ with "**quarter-comma meantone temperament**." This Renaissance tuning sets the "major third" as acoustically perfect. But the composition had to be chosen, and the trumpet had to transpose by 3 half-notes, to get the key the organ needed. In tour after the performance we spotted the extra split key on the console. D sharp and E flat need different, special, organ pipes!

Twelve Tone Equal temperament TET assigns twelfth root of two to all half-notes intervals to get perfect octaves. TET sounds equivalently good in all keys, but "lacks the spice and flavor of other systems." Music was re-purposed: centered on modulations, composer's needs, and instrument standardization. Bach, Mozart, and Beethoven score spectacular success! Physicist Helmholtz lead a groundswell of discontent with TET's loss of resonances and overtones, especially the major third.
Musicians strike beyond TET: Cellist Pablo Casals p155 taught "expressive intonation" intentionally playing "out of tune" to recapture major thirds beauty. Unaccompanied choirs also go out of tune to prioritize resonances and overtones. Guarneri Quartet p74 carefully sharpens notes while avoiding repetitions to avoid a "sterile and static" equal intonation. Bach craftily ducked TET with his 48 Klavier pieces which are apparently **"well- but not so even-"** tempered.

Conclusion: Musicianship remains subtle, reaching beyond the genius platforms of Pythagoras and Twelve Tone Equal Temperament TET

Musical Instruments and Twelve Tone Equal Temperament

Why was I intrigued as child in this esoteric subject? Why should anyone else be?

- As a child taking piano lessons: fascinated that "TET" equal temperament forces all keys "equivalent"   C vs. C sharp vs. F sharp
  - **But:** musical notation mysteriously super-redundant
  - **But:** Composers passionate about particular keys: Beethoven Quartet C sharp minor
  - **But:** String players & singers rebel from TET if not "enforced" by keyboard
- TET forces a perfect "octave;" all other intervals and resonances "muddled"
  - **"Just Intonation patch: smooth chords, melody notes that sound out of tune.**
    Equal Temperament – melody notes sound in tune, chords sound rough"
- Math: Ancient 530BC clash - perfect ratios abhor irrational numbers!
- Musical Instrument Cleverness – tape measure our piano & harp

NET: Musicianship dilemma made plain by math in these pages!

Bear with it, musicians: this ***arithmetic*** perspective is insightful!

2nd to last page: sharing the exciting moment when Pythagoras suddenly became simple genius!

Acknowledgments

- My wife, Amelia Parks
  - Masters in Music, Mills College   Suzuki Voice Teacher Trainer
  - Idea of this class obvious to her for decades: that singers and string players can reach past TET when a keyboard is not present. By adjusting adjust pitch slightly, singers and string players regain rich chords, resonances, and overtones lost by TET.
- Cellist Daughter Greta Parks, graduate Eastman School of Music, Boulder
  - Easily places musical theory perspective in physics terms that I can understand
  - Ideas of this this presentation also obvious to her & well-known to musicians
- House full of books - 1986 David Barbers "Bach, Beethoven, and the Boys"
  - Comical version of the anti-modernist 1610 Artusi clash with composer Monteverdi
- Russ and Tom Blackadar, school friends
  - Major Physicist Helmholtz "Sensations of Tone" Natural Tuning advocate - clash with Bach, …
  - Pointed out wonderful 2007 Ross Duffin "**How Equal Temperament Ruined Harmony**"

Pythagorean Comma: ca. 1% Arithmetic Mismatch
Spiral Instead of Circle of Fifths

1% of A440 is 4 beats/second – **nasty** for listeners

Pythagorean Perfection: Resonances and Overtones

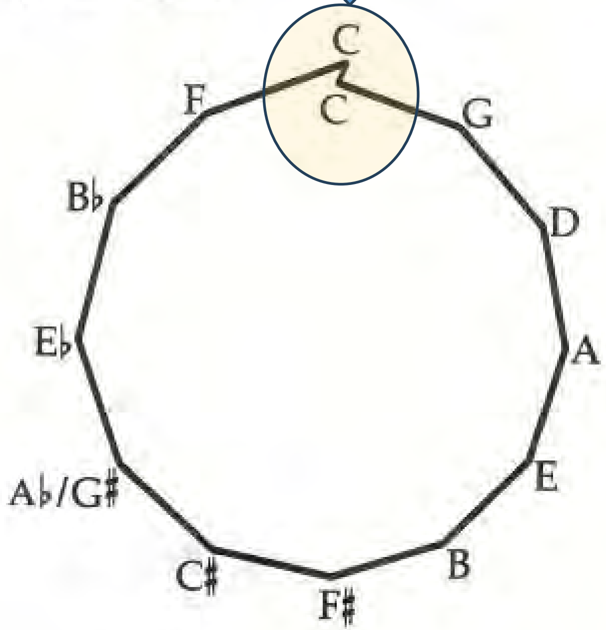
From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007 p25

Note to mathophobes: This is not math, it's arithmetic.

$$\frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = 129.746$$

$$\frac{2}{1} \times \frac{2}{1} \times \frac{2}{1} \times \frac{2}{1} \times \frac{2}{1} \times \frac{2}{1} \times \frac{2}{1} = 128.0$$

Figure 3. "Circle" of Pure Fifths.



(129.746-128) / 128 = 1.4%

Musical Instruments and TET or "Much Ado about 1.4%"

Equal TET and Just Temperament

https://hotrodharmonicas.com/equal-temperament-vs-just-intonation-war-of-the-musical-worlds/

Just Intonation: smooth chords, melody notes that sound out of tune.

Equal Temperament – melody notes sound in tune, chords sound rough.

Temperaments - cope with underlying arithmetic mismatch problem

Pythagorean and Just Temperament: 500BC origins China some analogies

Musicians around world connect with resonances & overtones

TET Equal Temperament, Bach era, irrational twelfth root

TET Revolution: scrap nice integer ratios at very center of Pythagoras great vision

Equal and Just Temperament

https://successmusicstudio.com/whats-the-difference-between-just-intonation-and-equal-temperament/

One disadvantage of ET revolves around the fact that it has fewer notes than JI.

•In Just Intonation, the notes have been tuned so that it creates better sounding harmonies.

•This means that JI can, in theory, have an infinite number of notes.

•Each JI note has a distinct frequency that resonates with the other chord tones to create a strikingly rich sound.

In contrast, ET has only 12 notes, which sound barely in tune when played together with each other.

•ET's harmonies don't resonate as well as they do in JI, because

ET has sonically compromised major thirds.

•As a result,

the ET chords will have beating and weak tone color, which can diminish your audience's listening pleasure.

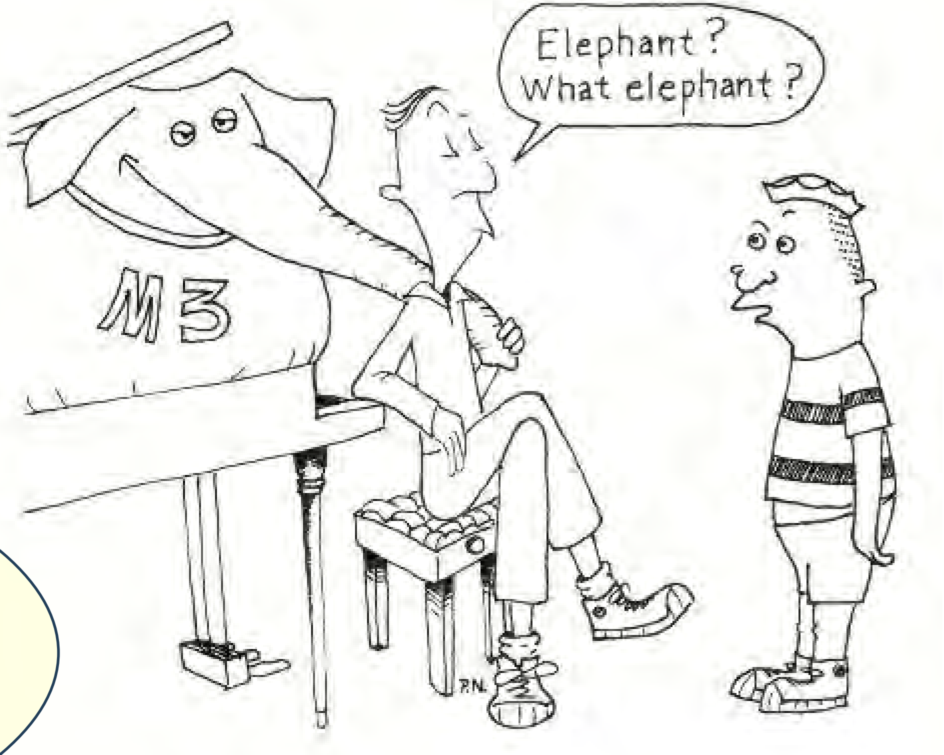
TET: Major 3d is "awful" in Equal Temperament

Just Tuning – major 3d is great, at cost of the bad "wolf"

All Tunings – do pretty well for fifths and fourths

From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007 p28

So convenient is this system that many musicians today don't notice how horrible the next important harmonic interval is in ET. The next simplest ratio after the 4:3 fourth belongs to the major third, at 5:4. The fifth and fourth of ET aren't bad, being out of acoustical purity by only about one-fiftieth of a semitone, but the major third is where ET fails the harmonic purity test. ET major thirds are extremely wide—about one-seventh of a semitone wider than acoustically pure 5:4 major thirds. That's about seven times the amount of discrepancy shown between ET fifths and acoustically pure fifths. This interval is the invisible elephant in our musical system today. Nobody notices how awful the major thirds are. Nobody comments. Nobody even recognizes that the



Ignoring the mammoth Major 3rd of ET

Out of Tune

https://musicinfo.io/blog/out-of-tune

Out of Tune

https://musicinfo.io/blog/out-of-tune   2022-08-30

The problem with Just Intonation

… Below you can listen and compare the differences in the sound of chords in 12 TET and Just Intonation and hear what happens if you try to play chords in Just Intonation outside the key they were tuned to.

Example 6. C6 12 TET
C6 is a chord of 1 3 5 6 or C E G A

Example 7. C6 just intonation

Example 8. C7, B7 and E7 in just intonation for C
C7 is a chord 1 3 5 7   C E G and B

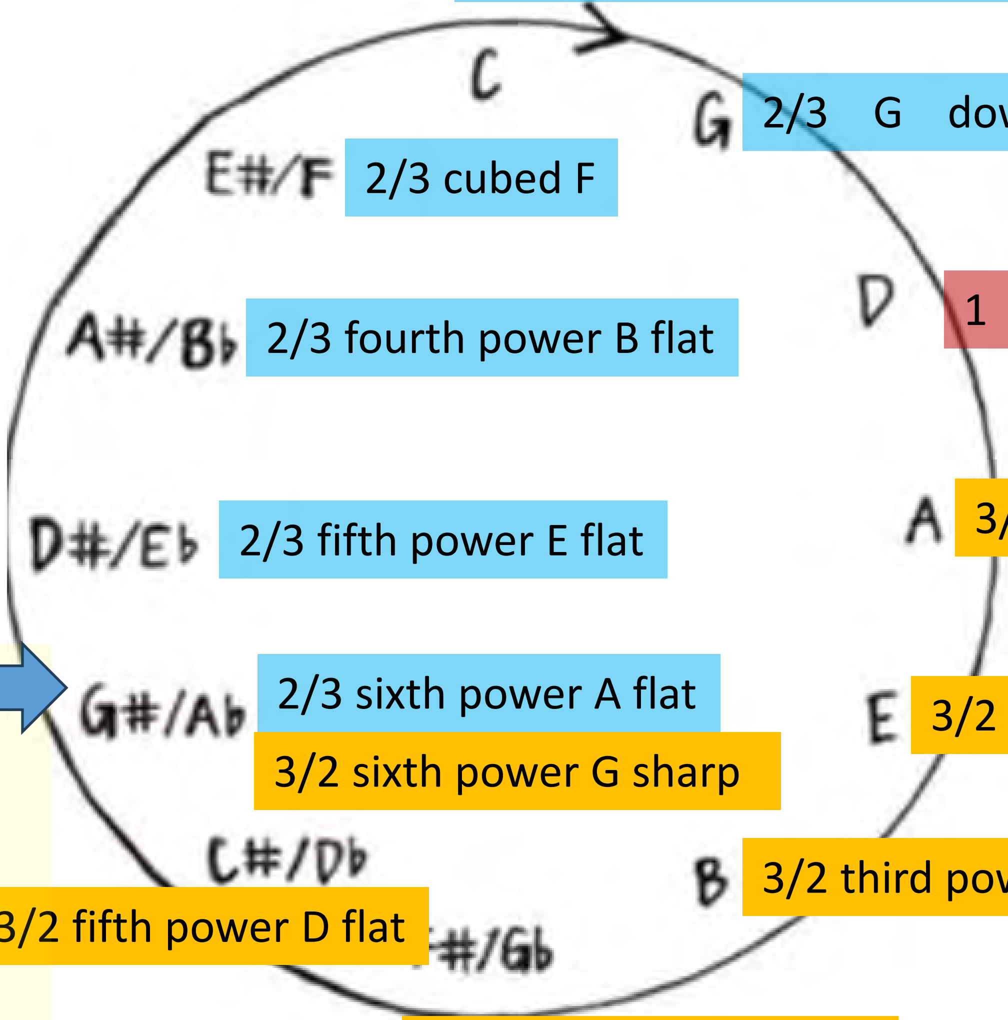
Purpose of examples
C6 and C7 should be good sounding in examples 6, 7, 8
Example 8   B7 or E7 should sound horrible in Just Intonation

Unpacking Pythagoras

- If you’re looking at it correctly, Pythagoras suddenly becomes simple
- I suddenly saw it correctly mid-summer: tremendously exciting, would like to share this excitement.

Circle of Fifths

Powers of 3/2 or 2/3 Pythagoras



2/3 squared C down two fourths

2/3 G down a fourth **293.33 x 2/3**

2/3 cubed F

1 pick D as root **293.33 beats/s**

2/3 fourth power B flat

2/3 fifth power E flat

3/2 A up by a fifth **293.33 x 3/2 = 440**

2/3 sixth power A flat

3/2 sixth power G sharp

3/2 second power E **293.33 x 3/2 x 3/2**

3/2 third power B

3/2 fifth power D flat

3/2 fourth power G flat

Our Famous "Comma"
A flat "Diminished fifth"
412.03 setting A440
G sharp "Augmented Fourth"
417.65 setting A440

SWEDEN "Just" organ - two keys
TET keyboards – skip 2nd key

Octaves – sound wave length at one meter or 347 beats/second, or half meter (octave higher) or one quarter meter (four octaves higher)

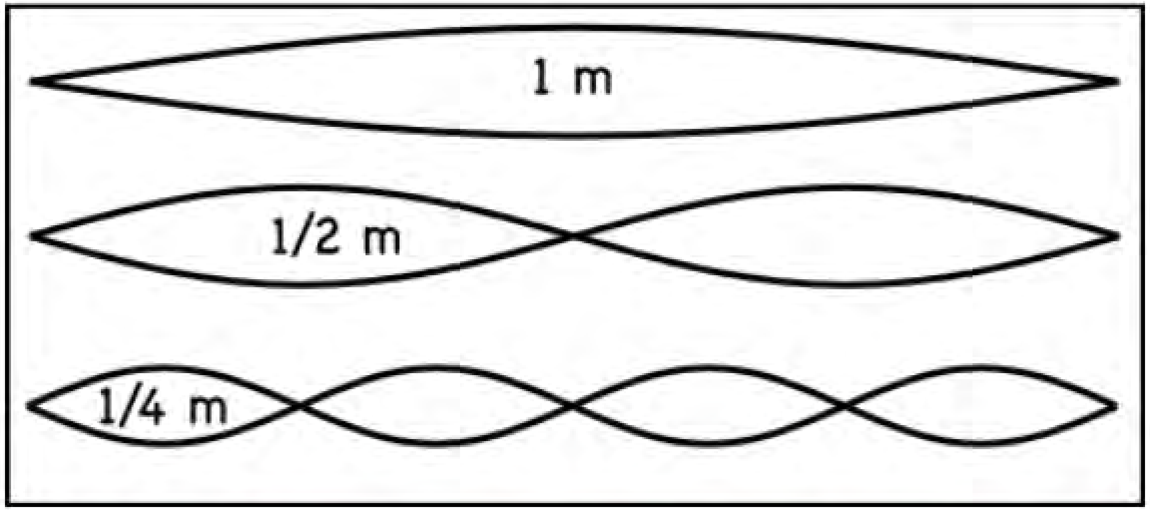
https://montessorimuddle.org/2012/04/16/octave-sound-samples/ April 2012

This "note" is a sound wave with a frequency (pitch) of 347 cycles per second (347 Hz), which has a wavelength of approximately 1 meter. It sounds like this.

Interesting to think of the **1 meter wavelength**,
and the size of concert halls,
and maybe the effect of shifting one's seat

Gregorian Chant immensely exploited sound reflection off of monastery or cathedral walls

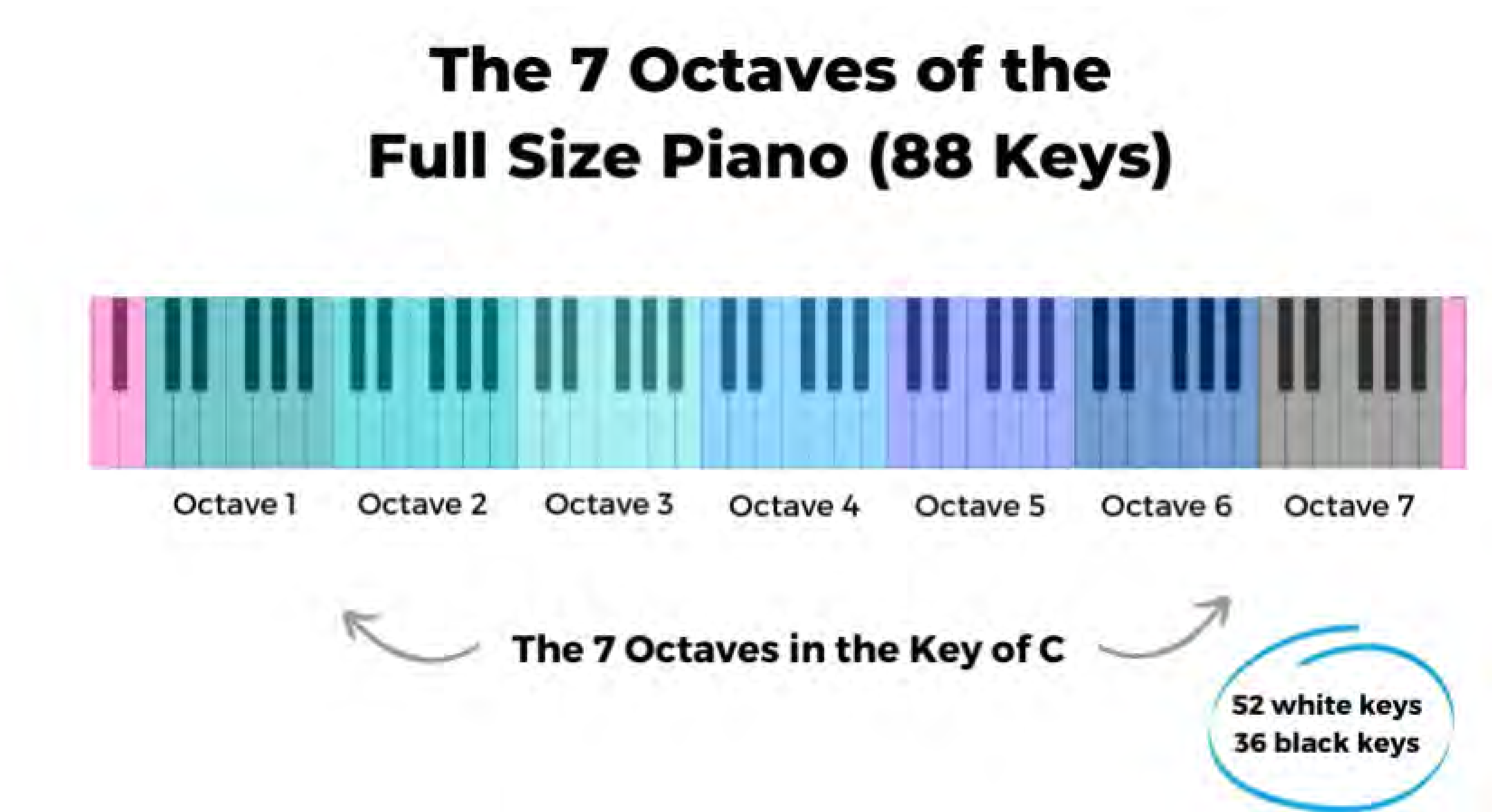
If one note has twice the frequency of the other, they're said to be one octave apart. For example, click on the image below to listen to the same note at different octaves:



Click the waves to hear the different octaves. The wavelengths of the sounds are shown (in meters).

Piano Keyboard – seven plus octaves

https://zinginstruments.com/how-many-octaves-on-a-piano/

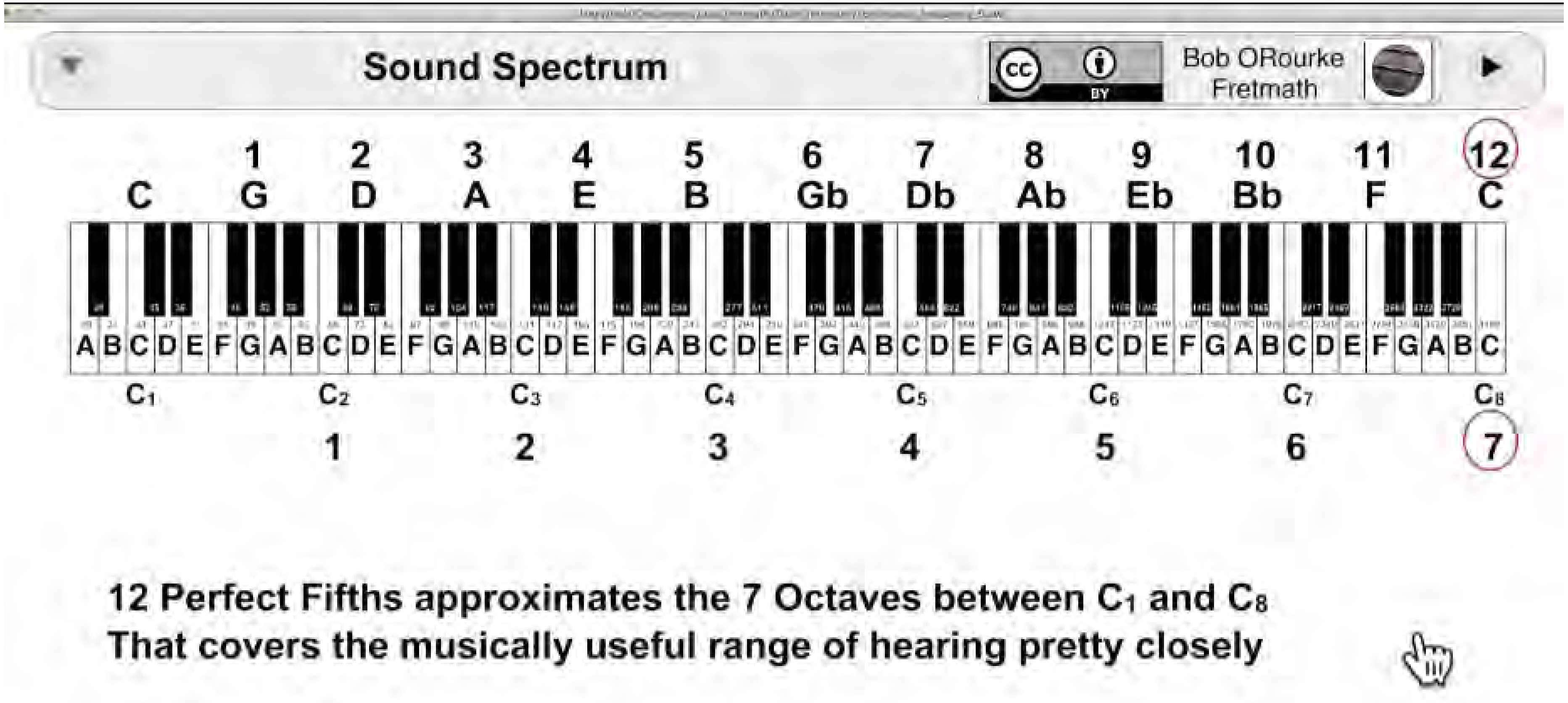


Circle of Twelve Fifths – Seven Octaves of the Piano

Twelve Notes C, G, D, A … G flat … C – get all twelve notes of the scale!

→Pythagoras 6th Century BC Great Discovery: calculate all note frequencies with powers of 3/2 and 2

https://www.pinterest.com/pin/750130881664413636/



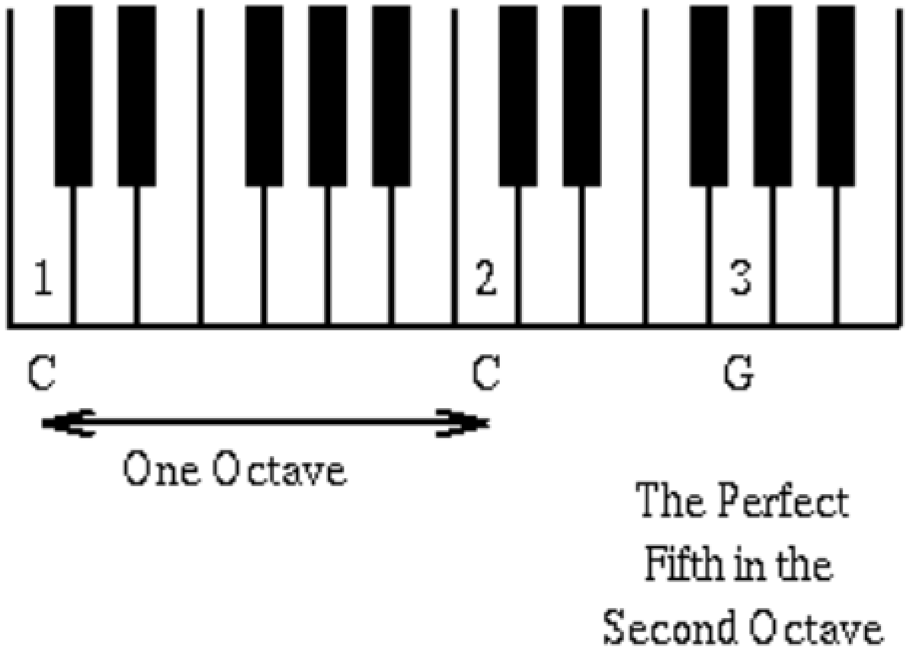
Pianos and Continued Fractions

Edward G. Dunne American Mathematical Society

https://oeis.org/DUNNE/TEMPERAMENT.HTML

There are two pieces of acoustics that matter now:

1. Going up one octave doubles the frequency. Thus, the C one octave up from middle C has a frequency of **2**.
2. Tripling the frequency moves to the perfect fifth in the next octave. In our case, this means that the **G** in the next octave has a frequency of **3**.

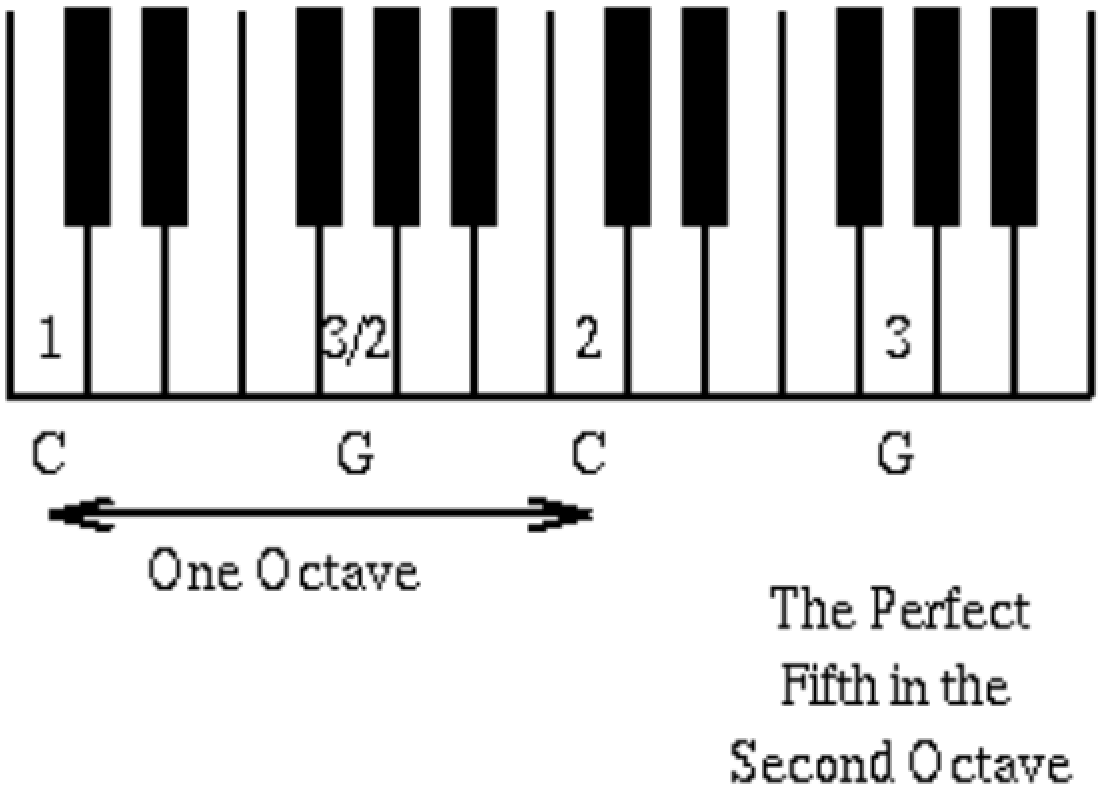


Pianos and Continued Fractions

Edward G. Dunne   American Mathematical Society

https://oeis.org/DUNNE/TEMPERAMENT.HTML

By inverting the rule that says that the note one octave than another must have double the frequency, we can fill-in the perfect fifth in the first octave. It should have half the frequency of the G in the second octave.

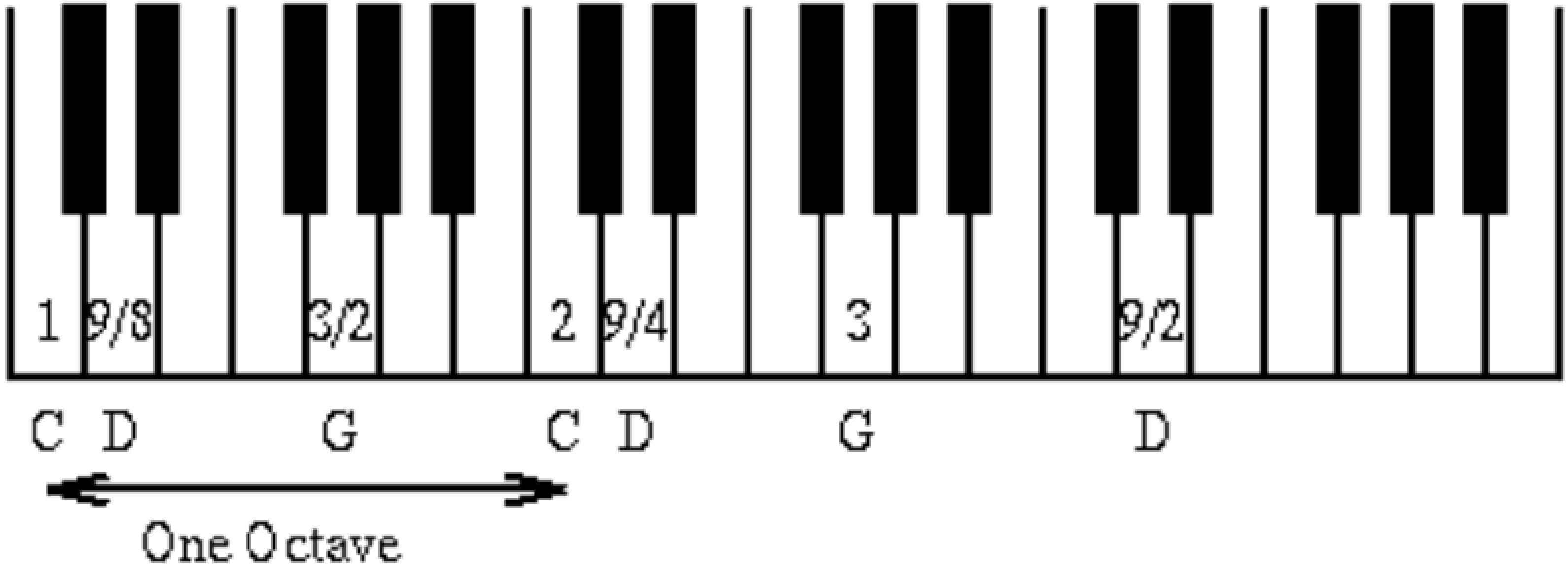


Pianos and Continued Fractions

Edward G. Dunne   American Mathematical Society

https://oeis.org/DUNNE/TEMPERAMENT.HTML

Following Pythagoras, we can now attempt to use these two rules to construct `all the notes', i.e., a complete chromatic scale.
The perfect fifth in the key of G is D. Thus we have, by tripling then halving, then halving again:

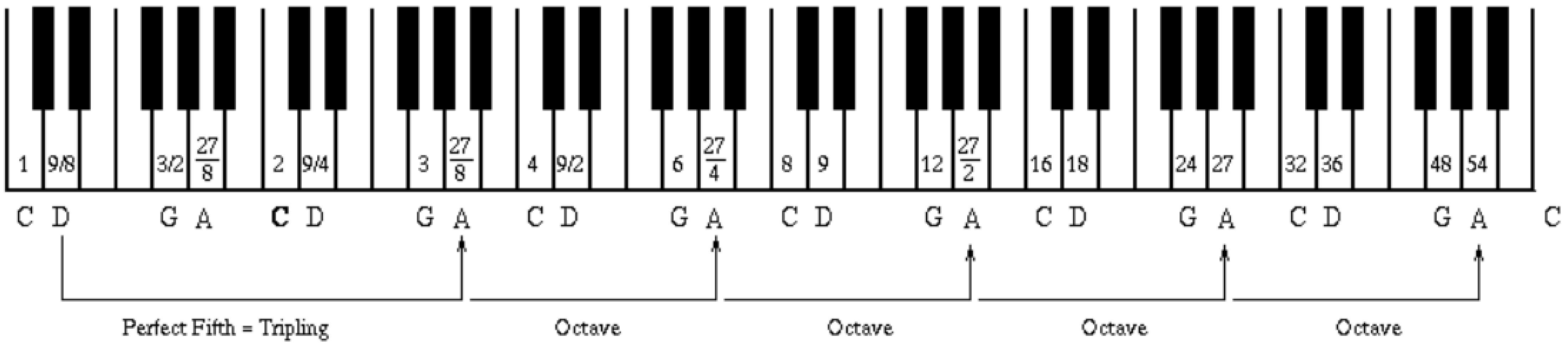


Pianos and Continued Fractions

Edward G. Dunne   American Mathematical Society

https://oeis.org/DUNNE/TEMPERAMENT.HTML

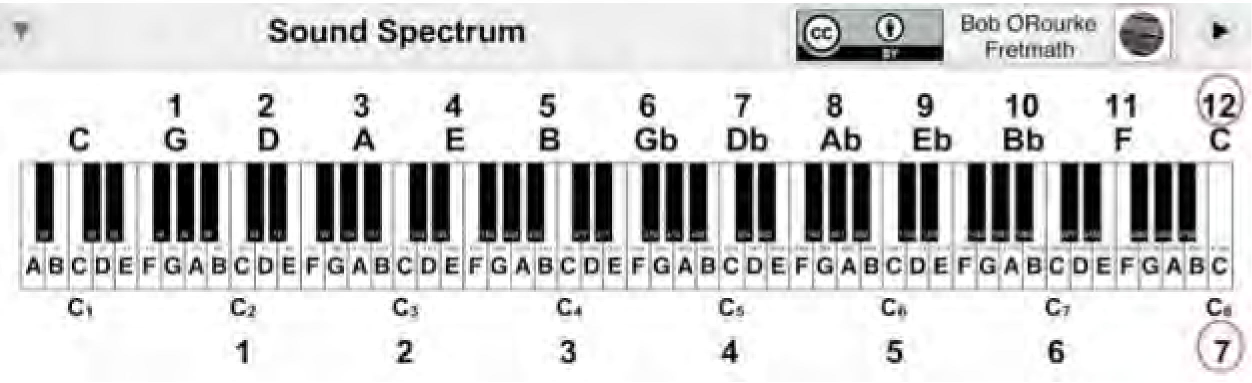
Repeat again: the perfect fifth in the key of D is A:

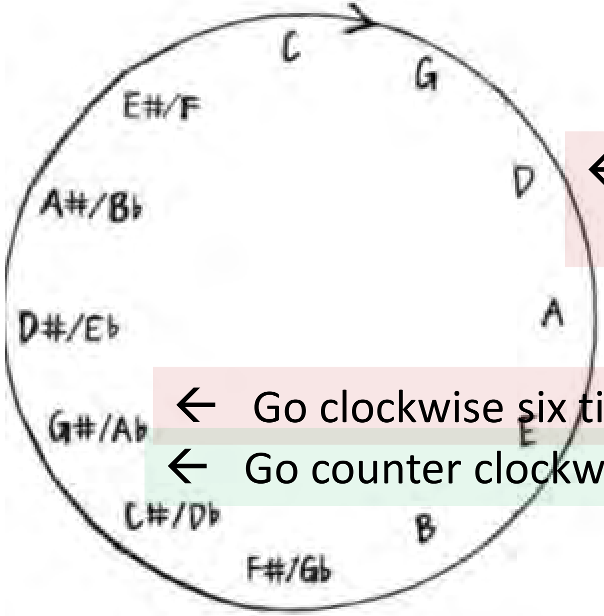


Following Pythagoras repeatedly: frequency doubling for octaves; frequency 3/2 for each fifth (clockwise circle of fifths) Eventually get frequencies for all twelve notes of the scale!

Pythagoras Music Base 12 System : One of ten greatest Math discoveries of the ancient world!

Clockwise on Circle of fifths – multiply by 3/2
Counterclockwise – divide by 3/2 each time





← Pick "D" as starting note

← Go clockwise six times to get G sharp

← Go counter clockwise six times for A flat

Clockwise vs. Counter clockwise from D
 Multiply or divide by sixth power of 3/2 to go half circle
 → Discover G sharp slightly different from A flat !!!

Pythagoras or Just Temperaments:
 You have discovered our infamous discrepancy or Comma or Wolf at G sharp versus A flat

Equal Temperament:
 Disguises the discrepancy but at a cost of losing resonant & overtone beauty particularly for the musical "third"
 Every note except the octave slightly "out of tune"

Pianos and Continued Fractions

Edward G. Dunne   American Mathematical Society

https://oeis.org/DUNNE/TEMPERAMENT.HTML

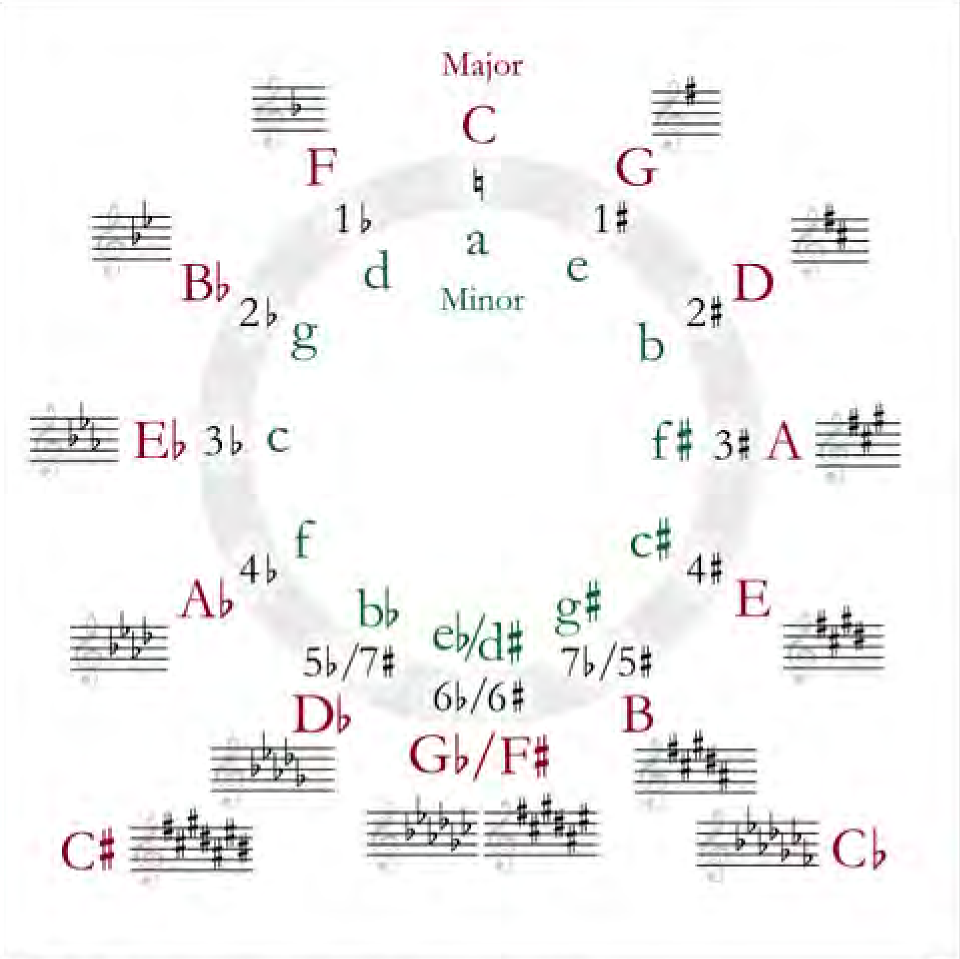
If we use the rule of doubling/halving for octaves, we arrive at the following frequencies for the twelve notes in our basic octave:

Frequency	Tonic				Fifth			Tonic
1	**C**	D	E	F	**G**	A	B	**C**
3 / 2	**G**	A	B	C	**D**	E	F#	**G**
9 / 8	**D**	E	F#	G	**A**	B	C#	**D**
27 / 16	**A**	B	C#	D	**E**	F#	G#	**A**
81 / 64	**E**	F#	G#	A	**B**	C#	D#	**E**
243 / 128	**B**	C#	D#	E	**F#**	G#	A#	**B**
729 / 512	**F#**	G#	A#	B	**C#**	D#	E# = F	**F#**
2187 / 1024	**C#**	D#	E# = F	F#	**G#**	A#	B# = C	**C#**
6561 / 4096	**G#**	A#	B# = C	C#	**D#**	E# = F	G	**G#**
19683 / 8192	**D#**	E# = F	G	G#	**A#**	B# = C	D	**D#**
59049 / 32768	**A#**	B# = C	D	D#	**E# = F**	G	S	**A#**
177147 / 131042	**E# = F**	G	A	A#	**B# = C**	D	E	**E# = F**
531441 / 262144	**C**	D	E	F	**G**	A	B	**C**

Circle of Fifths

https://en.wikipedia.org/wiki/Circle_of_fifths

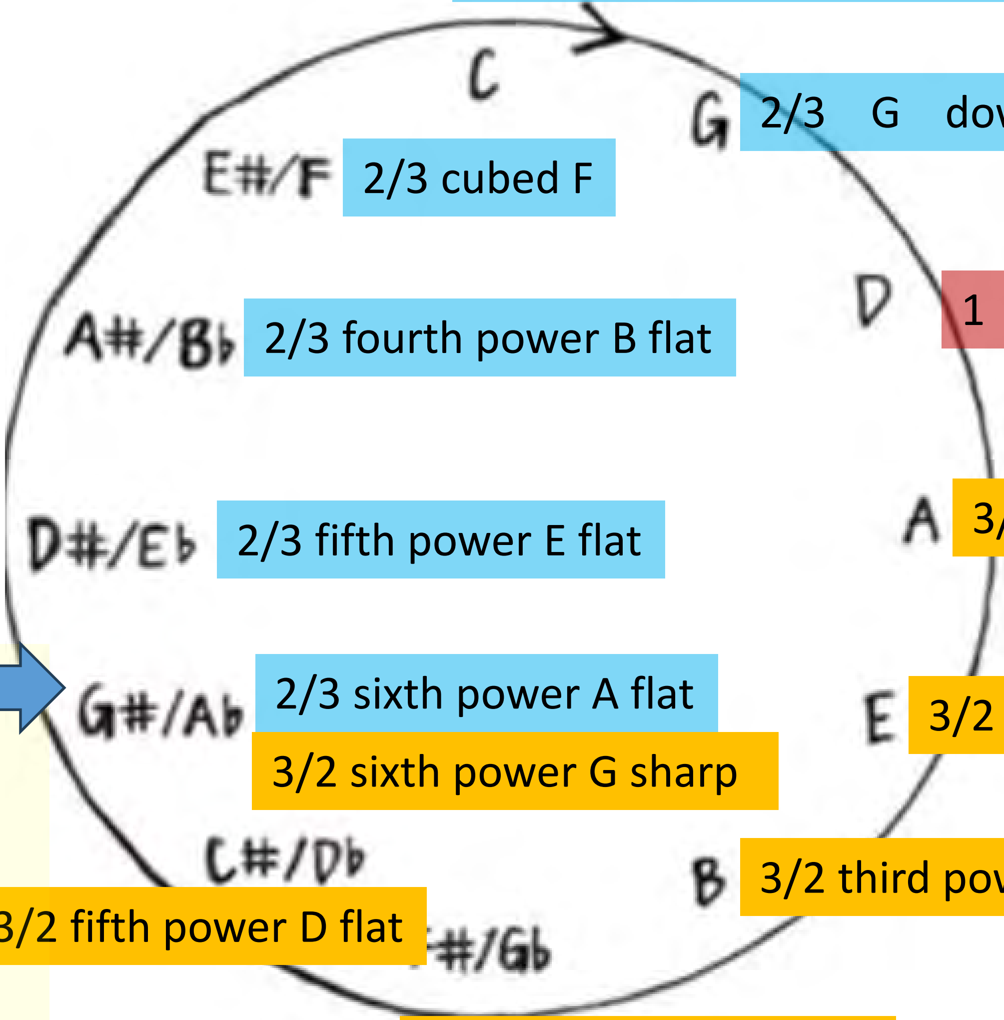
In music theory, the **circle of fifths** (or **circle of fourths**) is the relationship among the 12 tones (or pitches) of the chromatic scale, their corresponding key signatures, and the associated major and minor keys. More specifically, it is a geometrical representation of relationships among the 12 pitch classes of the chromatic scale in pitch class space.



Circle of fifths showing major and minor keys

Circle of Fifths

Powers of 3/2 or 2/3 Pythagoras



Our Famous "Comma"

A flat "Diminished fifth"

412.03 setting A440

G sharp "Augmented Fourth"

417.65 setting A440

SWEDEN "Just" organ - two keys

TET keyboards – skip 2nd key

Pythagorean Tuning

Fantastically Systematic

https://en.wikipedia.org/wiki/Pythagorean_tuning

Pythagoras:

Up by successive fifths
Powers of 3/2

Down by successive fourths
Powers of 2/3

“Normalize” within an octave
Dividing by powers of two

Seemingly crazy ratios make perfect sense!

No accident that millennia later, Pythagoras matches TET to within 0.5-1%

Note	Interval from D	Formula
D	unison	$\frac{1}{1}$
E♭	minor second	$\left(\frac{2}{3}\right)^5 \times 2^3$
E	major second	$\left(\frac{3}{2}\right)^2 \times \frac{1}{2}$
F	minor third	$\left(\frac{2}{3}\right)^3 \times 2^2$
F♯	major third	$\left(\frac{3}{2}\right)^4 \times \left(\frac{1}{2}\right)^2$
G	perfect fourth	$\frac{2}{3} \times 2$
A♭	diminished fifth	$\left(\frac{2}{3}\right)^6 \times 2^4$
G♯	augmented fourth	$\left(\frac{3}{2}\right)^6 \times \left(\frac{1}{2}\right)^3$
A	perfect fifth	$\frac{3}{2}$
B♭	minor sixth	$\left(\frac{2}{3}\right)^4 \times 2^3$
B	major sixth	$\left(\frac{3}{2}\right)^3 \times \frac{1}{2}$
C	minor seventh	$\left(\frac{2}{3}\right)^2 \times 2^2$
C♯	major seventh	$\left(\frac{3}{2}\right)^5 \times \left(\frac{1}{2}\right)^2$

Pythagorean Tuning

Fantastically Systematic

https://en.wikipedia.org/wiki/Pythagorean_tuning

Pythagoras:

Up by successive fifths
Powers of 3/2

Down by successive fourths
Powers of 2/3

"Normalize" within an octave
Dividing by powers of two

Seemingly crazy ratios make perfect sense!

No accident that millennia later, Pythagoras matches TET to within 0.5-1%

Note	Interval from D	Formula
D	unison	$\frac{1}{1}$
E♭	minor second	$\left(\frac{2}{3}\right)^5 \times 2^3$
E	major second	$\left(\frac{3}{2}\right)^2 \times \frac{1}{2}$
F	minor third	$\left(\frac{2}{3}\right)^3 \times 2^2$
F♯	major third	$\left(\frac{3}{2}\right)^4 \times \left(\frac{1}{2}\right)^2$
G	perfect fourth	$\frac{2}{3} \times 2$
A♭	diminished fifth	$\left(\frac{2}{3}\right)^6 \times 2^4$
G♯	augmented fourth	$\left(\frac{3}{2}\right)^6 \times \left(\frac{1}{2}\right)^3$
A	perfect fifth	$\frac{3}{2}$
B♭	minor sixth	$\left(\frac{2}{3}\right)^4 \times 2^3$
B	major sixth	$\left(\frac{3}{2}\right)^3 \times \frac{1}{2}$
C	minor seventh	$\left(\frac{2}{3}\right)^2 \times 2^2$
C♯	major seventh	$\left(\frac{3}{2}\right)^5 \times \left(\frac{1}{2}\right)^2$

TET	Pyth	Delta
293.66	293.33	0.33 hz
311.13	309.02	-2.1
329.63	330.00	0.37
349.23	347.65	-1.6
369.99	371.24	1.25
392	391.11	-0.89
Comma	412.03	-3.9
415.93	417.65	1.72
440	440	Ref
466.16	463.53	-2.6
493.88	494.99	1.1
523.25	521.48	-1.8
554.37	556.87	2.5

Pythagorean Tuning

Fantastically Systematic

https://en.wikipedia.org/wiki/Pythagorean_tuning

Pythagoras:

Up by successive fifths
Powers of 3/2

Down by successive fourths
Powers of 2/3

"Normalize" within an octave
Dividing by powers of two

Seemingly crazy ratios make perfect sense!

No accident that millennia later, Pythagoras matches TET to within 0.5-1%

Note	Interval from D	Formula	=	=	Frequency ratio	Size (cents)	12-TET-dif (cents)
D	unison	$\frac{1}{1}$	$3^0 \times 2^0$	$\frac{3^0}{2^0}$	$\frac{1}{1}$	0.00	0.00
E♭	minor second	$\left(\frac{2}{3}\right)^5 \times 2^3$	$3^{-5} \times 2^8$	$\frac{2^8}{3^5}$	$\frac{256}{243}$	90.22	−9.78
E	major second	$\left(\frac{3}{2}\right)^2 \times \frac{1}{2}$	$3^2 \times 2^{-3}$	$\frac{3^2}{2^3}$	$\frac{9}{8}$	203.91	3.91
F	minor third	$\left(\frac{2}{3}\right)^3 \times 2^2$	$3^{-3} \times 2^5$	$\frac{2^5}{3^3}$	$\frac{32}{27}$	294.13	−5.87
F♯	major third	$\left(\frac{3}{2}\right)^4 \times \left(\frac{1}{2}\right)^2$	$3^4 \times 2^{-6}$	$\frac{3^4}{2^6}$	$\frac{81}{64}$	407.82	7.82
G	perfect fourth	$\frac{2}{3} \times 2$	$3^{-1} \times 2^2$	$\frac{2^2}{3^1}$	$\frac{4}{3}$	498.04	−1.96
A♭	diminished fifth	$\left(\frac{2}{3}\right)^6 \times 2^4$	$3^{-6} \times 2^{10}$	$\frac{2^{10}}{3^6}$	$\frac{1024}{729}$	588.27	−11.73
G♯	augmented fourth	$\left(\frac{3}{2}\right)^6 \times \left(\frac{1}{2}\right)^3$	$3^6 \times 2^{-9}$	$\frac{3^6}{2^9}$	$\frac{729}{512}$	611.73	11.73
A	perfect fifth	$\frac{3}{2}$	$3^1 \times 2^{-1}$	$\frac{3^1}{2^1}$	$\frac{3}{2}$	701.96	1.96
B♭	minor sixth	$\left(\frac{2}{3}\right)^4 \times 2^3$	$3^{-4} \times 2^7$	$\frac{2^7}{3^4}$	$\frac{128}{81}$	792.18	−7.82
B	major sixth	$\left(\frac{3}{2}\right)^3 \times \frac{1}{2}$	$3^3 \times 2^{-4}$	$\frac{3^3}{2^4}$	$\frac{27}{16}$	905.87	5.87
C	minor seventh	$\left(\frac{2}{3}\right)^2 \times 2^2$	$3^{-2} \times 2^4$	$\frac{2^4}{3^2}$	$\frac{16}{9}$	996.09	−3.91
C♯	major seventh	$\left(\frac{3}{2}\right)^5 \times \left(\frac{1}{2}\right)^2$	$3^5 \times 2^{-7}$	$\frac{3^5}{2^7}$	$\frac{243}{128}$	1109.78	9.78

TET	Pyth	Delta
293.66	293.33	0.33 hz
311.13	309.02	-2.1
329.63	330.00	0.37
349.23	347.65	-1.6
369.99	371.24	1.25
392	391.11	-0.89
Comma	412.03	-3.9
415.93	417.65	1.72
440	440	Ref
466.16	463.53	-2.6
493.88	494.99	1.1
523.25	521.48	-1.8
554.37	556.87	2.5

5-13-2022

The mathematical foundation of the musical scales andThe mathematical foundation of the musical scales and overtones **Michaela DuBose-Schmitt** Mississippi State University, michaela.duboseschmitt@gmail.com

https://scholarsjunction.msstate.edu/cgi/viewcontent.cgi?article=6428&context=td

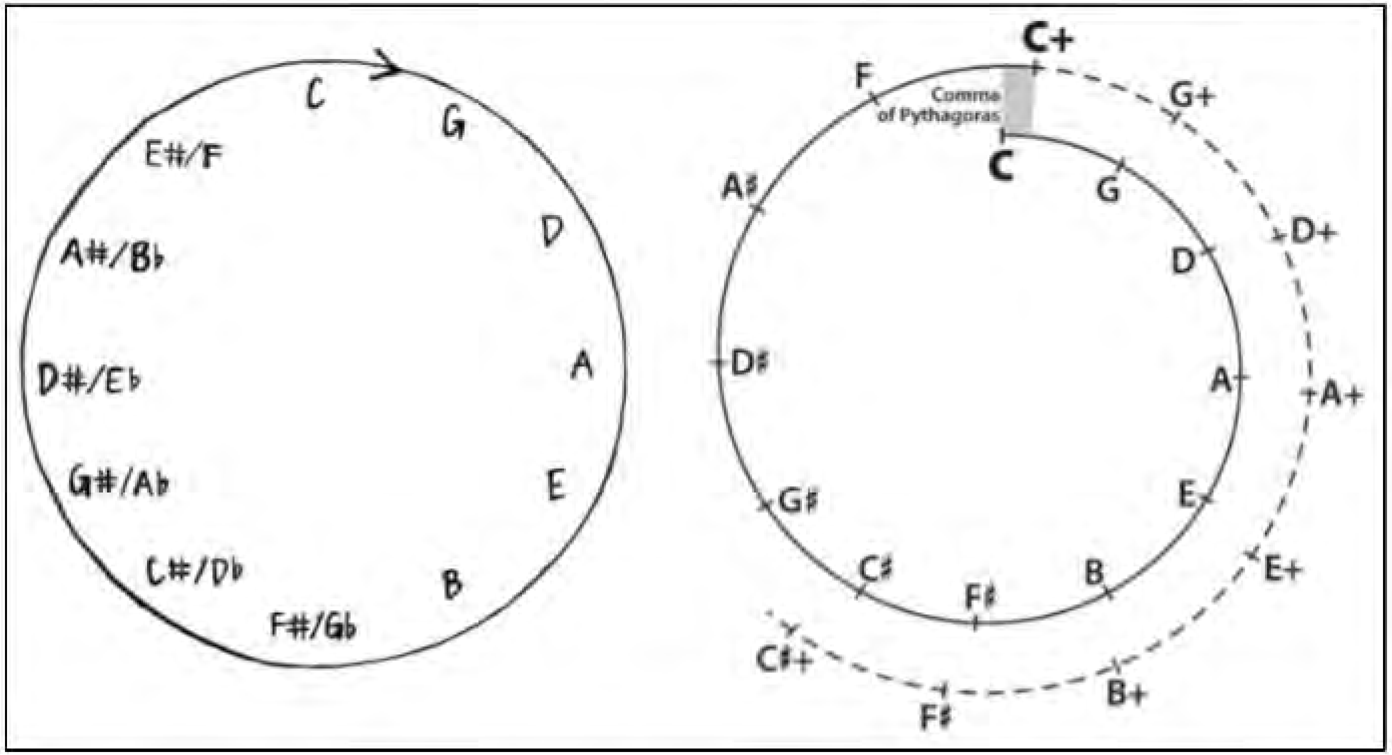


Figure 1.3 Comparison of the true circle of fifths against a Pythagorean circle of fifths [3]

The circle on the left shows a true circle of fifths in which the octaves line up; the "circle" on the right accounts for the Pythagorean comma and shows the discrepancy in ratio as the octaves increase.

Just & Pythagorean vs. Equal

Pythagorean & Just

Nice Integer ratios
Great sounding chords
Resonances & Overtones
"Wolf" Can't Modulate
Non-Western music systems too

TET

Radical Re-purposing of music
Twelve Tone Equal Temperament
Abandon nice ratios!
Pythagoras abhors twelfth root of two
Octave perfect frequency double
Major third especially "out of tune"
Freely modulate keys
Composers love flexibility

Principles of Musical Acoustics W Hartmann

Table D1 Notes of the scale in three temperaments

Interval	Just tuning	Pythagorean tuning	Equal temperament
Unison	1.	1.	$2^0 = 1.$
minor Second	16/15 = 1.067	256/243 = 1.053	$2^{1/12} = 1.059$
Major Second	10/9 = 1.111 or 9/8 = 1.125	9/8 = 1.125	$2^{2/12} = 1.122$
minor Third	6/5 = 1.200	32/27 = 1.185	$2^{3/12} = 1.189$
Major Third	5/4 = 1.250	81/64 = 1.266	$2^{4/12} = 1.260$
Fourth	4/3 = 1.333	4/3 = 1.333	$2^{5/12} = 1.335$
Tritone	45/32 = 1.406 or 64/45 = 1.422	1024/729 = 1.405 or 729/512 = 1.424	$2^{6/12} = 1.414$
Fifth	3/2 = 1.500	3/2 = 1.500	$2^{7/12} = 1.498$
minor Sixth	8/5 = 1.600	128/81 = 1.580	$2^{8/12} = 1.587$
Major Sixth	5/3 = 1.667	27/16 = 1.688	$2^{9/12} = 1.682$
minor Seventh	7/4 = 1.750 or 16/9 = 1.778 or 9/5 = 1.800	16/9 = 1.778	$2^{10/12} = 1.782$
Major Seventh	15/8 = 1.875	243/128 = 1.898	$2^{11/12} = 1.888$
Octave	2/1 = 2.000	2/1 = 2.000	$2^{12/12} = 2.000$

Book has scrubbed away the Octave 2.0 problem with Pythagoras!

Twelve Tone Equal Temperament 12-TET compared to "just intonation" ratios.

An octave is a perfect doubling; the fifth 3/2 and fourth 4/3 are extremely close to natural ratios
Major third is significantly off some say in TET

https://en.wikipedia.org/wiki/Equal_temperament and https://en.wikipedia.org/wiki/Musical_temperament

Comparison with Just Intonation [edit]

The intervals of 12-TET closely approximate some intervals in just intonation.[44] The fifths and fourths are almost indistinguishably close to just intervals, while thirds and are further away.

In the following table the sizes of various just intervals are compared against their equal-tempered counterparts, given as a ratio as well as cents.

Name	Exact value in 12-TET	Decimal value in 12-TET	Cents	Just intonation interval	Cents in just intonation	Difference
Unison (C)	$2^{0/12} = 1$	1	0	$1/1 = 1$	0	0
Minor second (C♯/D♭)	$2^{1/12} = \sqrt[12]{2}$	1.059463	100	$16/15 = 1.06666...$	111.73	−11.73
Major second (D)	$2^{2/12} = \sqrt[6]{2}$	1.122462	200	$9/8 = 1.125$	203.91	−3.91
Minor third (D♯/E♭)	$2^{3/12} = \sqrt[4]{2}$	1.189207	300	$6/5 = 1.2$	315.64	−15.64
Major third (E)	$2^{4/12} = \sqrt[3]{2}$	1.259921	400	$5/4 = 1.25$	386.31	+13.69
Perfect fourth (F)	$2^{5/12} = \sqrt[12]{32}$	1.334840	500	$4/3 = 1.33333...$	498.04	+1.96
Tritone (F♯/G♭)	$2^{6/12} = \sqrt{2}$	1.414214	600	$7/5 = 1.4$ $10/7 = 1.42857...$	582.51 617.49	+17.49 −17.49
Perfect fifth (G)	$2^{7/12} = \sqrt[12]{128}$	1.498307	700	$3/2 = 1.5$	701.96	−1.96
Minor sixth (G♯/A♭)	$2^{8/12} = \sqrt[3]{4}$	1.587401	800	$8/5 = 1.6$	813.69	−13.69
Major sixth (A)	$2^{9/12} = \sqrt[4]{8}$	1.681793	900	$5/3 = 1.66666...$	884.36	+15.64
Minor seventh (A♯/B♭)	$2^{10/12} = \sqrt[6]{32}$	1.781797	1000	$16/9 = 1.77777...$	996.09	+3.91
Major seventh (B)	$2^{11/12} = \sqrt[12]{2048}$	1.887749	1100	$15/8 = 1.875$	1088.27	+11.73
Octave (C)	$2^{12/12} = 2$	2	1200	$2/1 = 2$	1200.00	0

MAJOR THIRD
Problem interval
for TET

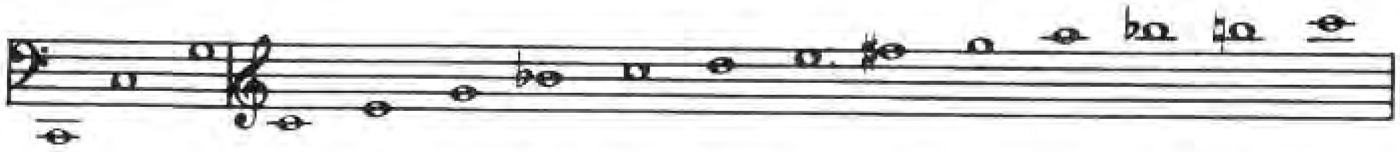
Perfect in Just

Octave
slight inherent mismatch
discovered 500BC
Commas, Wolf, …
Need for Temperaments

Fourth
and
Fifth
Good all
systems

Octave
set to 2
TET & Just

Harmonic Series: Inharmonic Partials



Pitch	C	C	G	C	E	G	B♭	C	D	E	F#	G	A	B♭	B	C
Partial No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Natural Spectrum of "C"	65.4	130.8	196.2	261.6	327	392.4	457.8	523.2	588.6	654	719.4	784.8	850.2	915.6	981	1046.4
Tempered Spectrum of "C"	65.4	130.8	196.0	261.6	329.6	392	466.1	523.2	587.4	659.2	698.4	783.9	880	932.4	987.8	1046.4
Beats per second					2		10		1	5	9		30	17	6	

Fig. 59. The Harmonic Series

Tempered Scale,
sounded with Natural Scale
of "C"

"Differences in frequences give
beats or a vibrato effect to
the sound rather than smoothness"

The inharmonic partials of the tempered scale shown above (5, 7, 9, 10, 11, 13, 15) are not exact multiples of the fundamental, and when sounded with the natural scale, they give the effect of roughness. The beats caused by the difference in frequency gives a vibrato effect to the sound rather than smoothness. Inharmonic partials are usually among the higher partials and may be small or large in amplitude.

110

D. Ralph Appelman, Science of Vocal Pedagogy, 1967

Choir and Voice

Good Choir directors:
add a "**just hear it**" element
Hear and search for Overtones
Prioritize resonance & overtones over TET frequencies
Allow carefully calibrated inconsistencies
(if there's no keyboard or orchestra)

The Pythagorean Scale and Just Intonation

https://mathcs.holycross.edu/~groberts/Courses/MA110/Lectures/PythScale-web.pdf

22Mar2018 Gareth E R

Figure: Jamming out on the wrenchophone at the Peabody Essex Museum. The ratios of the weights of the wrenches are small integer ratios like 2 : 1, 3 : 2, and 4 : 3.

Trumpet Organ Concert in Stockholm – June 2019 church with two organs

Trumpet Organ Concert in Stockholm – June 2019 German Church Gamla Stan, church with two organs

”Our” Sweden 1651 *Düben* Organ
“quarter-comma meantone
with sub semitones for E flat and D sharp”

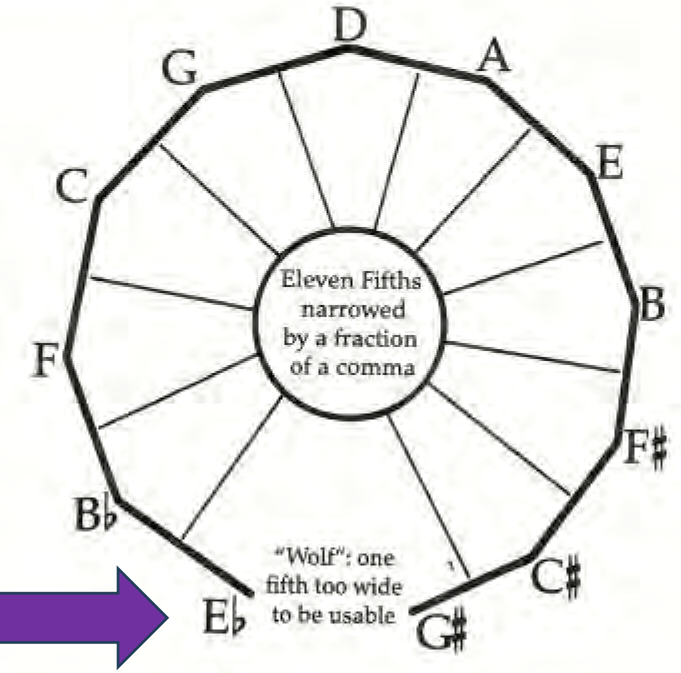
Quarter Comma Meantone – Our Sweden Organ !

From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007 p35

Quarter Comma Meantone – Renaissance
All the fifths but one **"Wolf"** tempered the same amount

Just Temperament: Good whole number ratios!
Great resonances, overtones!
Achieves Fantastically good "third"
Can't Modulate – crippling for post Bach composers
Unlike extremely poor "third" for TET

Figure 6. Regular Meantone Fifth "Circle."



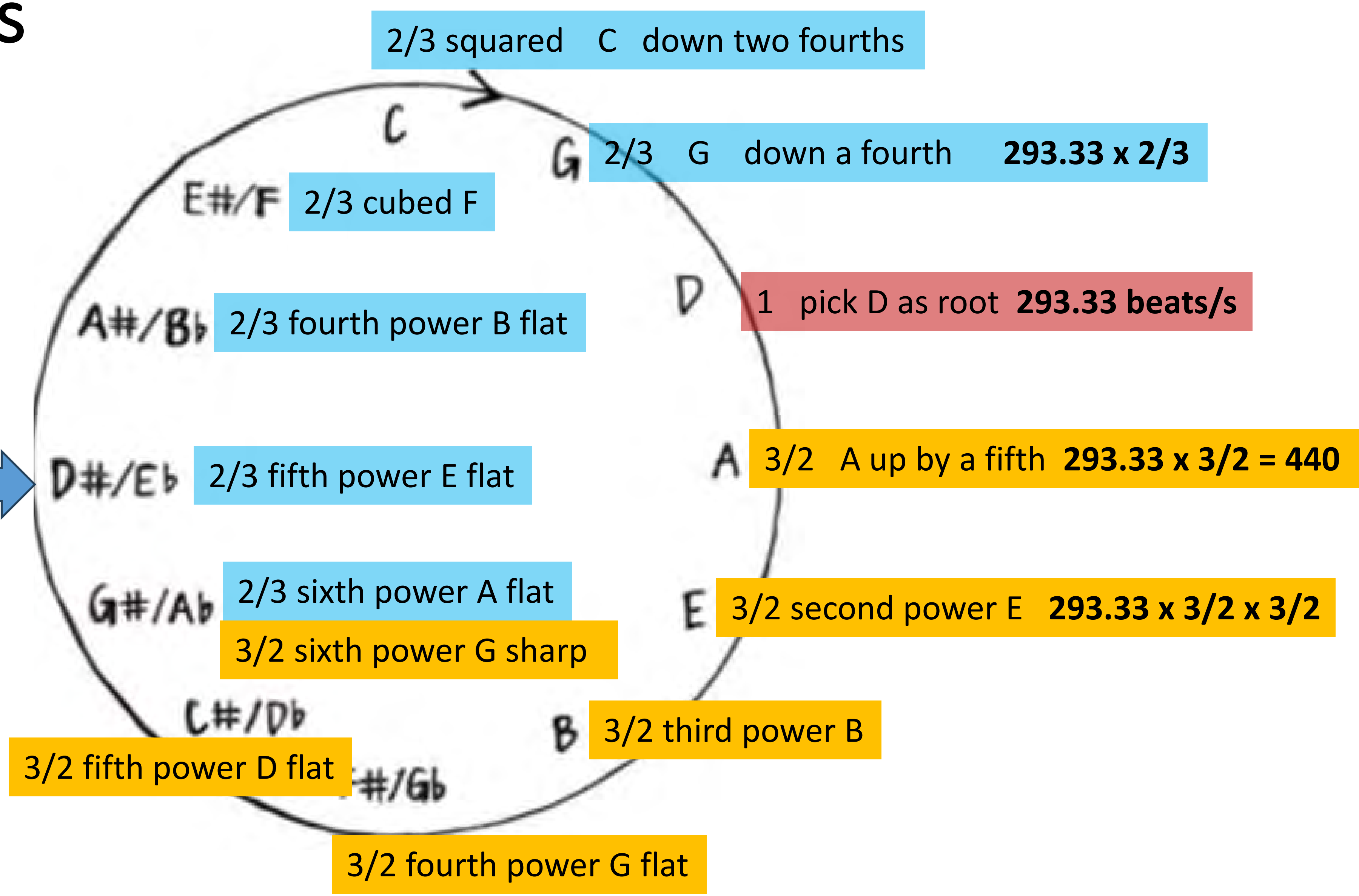
"Our" Sweden 1651 *Düben* Organ "quarter-comma meantone with sub semitones for E flat and D sharp"

While this tuning system continued in common use in many places until around the end of the seventeenth century, already t

Circle of Fifths

Powers of 3/2 or 2/3 Pythagoras

”Our” Sweden 1651 *Düben* Organ “quarter-comma meantone with sub semitones for E flat and D sharp”



Trumpet Organ Concert in Stockholm – church with two organs:

1651 *Düben* Organ “quarter-comma meantone”

1887-Romantic Style

Different tuning systems

the original organ in Overtorneå was restored.[29] The German Church in Stockholm now houses on its south wall the second replica of its former organ, inaugurated in 2004. It is tuned in quarter-comma meantone, with subsemitones for E♭/D♯.

JAN GUSTAVSSON (Motala)
TROMPETE

MICHAEL DIERKS (Stockholm)
DÜBEN-ORGEL (1651/2004) UND JUNO-ORGEL (1887)

The DÜBEN-ORGAN was originally built in 1608 on the gallery in the tower of the church. It was several times extensively modified and enlarged by different organ builders, until it was finally completed in 1651 and by this time one of Scandinavia's most magnificent instruments. Fallen out of fashion in the 18th century, it was sold to northern Sweden in 1779.

Today on the south gallery, there is a faithful replica of the once famous baroque organ of the German Church. It was built in 2004 by Grönlunds organ builders and is named after Andreas and Gustav Düben who were organists in St. Gertrud's Church during the time of Swedish hegemony in the 17th century, when the German Church was a center of European church music.

JUNO-ORGAN. After a fire in the tower of the German Church in 1878, a new organ was built in 1887 by the Swedish organ builders Åkerman & Lund in romantic style. When this instrument had fallen out of fashion (history sometimes repeats itself, even in the German church), it was stored in 1972 in a cellar in the residential quarter "Juno" in the Old town of Stockholm and replaced by an organ of inferior quality.

In 2018, however, it was possible to restore the preserved romantic organ in its original shape and to rebuild it at its original location in the tower of the German Church.

The work was carried out by organ builders Bergenblad & Jonsson and Åkerman & Lund. To pay tribute to the changeful history of this instrument, it was named "Juno-Organ".

Quarter Comma Meantone
(Sweden 1651 *Düben* Organ)

https://en.wikipedia.org/wiki/Quarter-comma_meantone

Quarter-comma meantone, or **1 / 4 -comma meantone**, was the most common meantone temperament in the sixteenth and seventeenth centuries, and was sometimes used later. In this system the perfect fifth is flattened by one quarter of a syntonic comma (81 : 80), with respect to its just intonation used in Pythagorean tuning (frequency ratio 3 : 2); the result is $3 / 2 \times [\, 80 / 81 \,]^{1/4} = \sqrt[4]{5} \approx 1.49535$, or a fifth of 696.578 cents. (The 12th power of that value is 125, whereas 7 octaves is 128, and so falls 41.059 cents short.) This fifth is then iterated to generate the diatonic scale and other notes of the temperament.

The purpose is to obtain justly intoned major thirds (with a frequency ratio equal to 5 : 4).

It was described by Pietro Aron in his *Toscanello de la Musica* of 1523, by saying the major thirds should be tuned to be "sonorous and just, as united as possible."[1] Later theorists Gioseffo Zarlino and Francisco de Salinas described the tuning with mathematical exactitude.

Irregular Temperament Fifth Circle

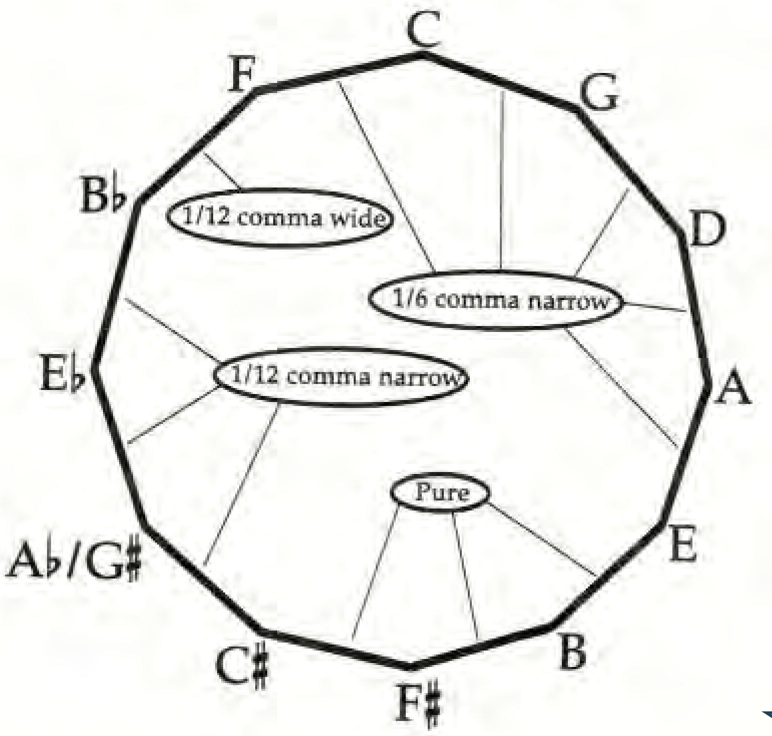
Lehman 2005 "Bach's Extraordinary Temperament: our Rosetta Stone" Early Music 33 (2005.) Bach proposed to encode an irregular temperament that gives the closed circle

From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007 p37 For Bach see p148, 169 Rosetta Stone footnote.
Also see https://en.wikipedia.org/wiki/The_Well-Tempered_Clavier B. Lehman (2004, 2005)[22] proposed a

1 /6 and 1/ 12 comma layout derived from Bach's loops

"Rosetta Stone" Proposal 2005 that
Bach 48 well-tempered pieces
were actually customized, irregular temperaments
and not TET !

Figure 7. Sample Irregular Temperament Fifth Circle.



What is a comma?

https://en.wikipedia.org/wiki/Comma_(music)

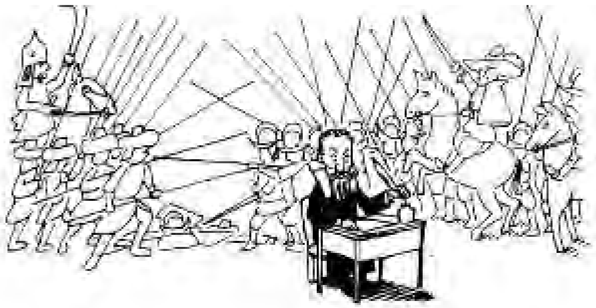
In music theory, a **comma** is a very small interval, the difference resulting from tuning one note two different ways.[1] Strictly speaking, there are only two kinds of comma, the syntonic comma, "the difference between a just major 3rd and four just perfect 5ths less two octaves", and the Pythagorean comma, "the difference between twelve 5ths and seven octaves".[2] The word *comma* used without qualification refers to the syntonic comma,[3] which can be defined, for instance, as the difference between an F♯ tuned using the D-based Pythagorean tuning system, and another F♯ tuned using the D-based quarter-comma meantone tuning system.

Within the same tuning system, two enharmonically equivalent notes (such as G♯ and A♭) may have a slightly different frequency, and the interval between them is a comma. For example, in extended scales produced with five-limit tuning an A♭ tuned as a major third below C_5 and a G♯ tuned as two major thirds above C_4 are not exactly the same note, as they would be in equal temperament. The interval between those notes, the diesis, is an easily audible comma (its size is more than 40% of a semitone).

Monteverdi 1610 Vespers Crucial

transitional figure between the Renaissance and Baroque periods of music history

Modernist music breaking rules



Monteverdi went along for the ride.

In 1600, a music theorist named Giovanni Artusi published an attack on modern music, which he thought was totally incomprehensible. Artusi complained that new composers were breaking all the established rules, laid down after centuries of noble tradition.[4] Although he didn't name Monteverdi specifically, it was pretty obvious to everyone whom he was complaining about.

Monteverdi didn't seem to mind; he went on composing just the same. A few years later, Artusi published another attack, this time naming names.

Monteverdi wasn't about to give in. Quite the contrary: He went so far as to develop a whole new style of music, which marked the beginning of opera. His music drama *Orfeo*, of 1607, could be considered the first true opera, although the idea had come from the earlier writings of Vincenzo Galilei (father of the astronomer Galileo), who was the champion of what he called the "representative style."

Monteverdi sneaks in
Even Temperament

Music theorist (Artusi) pounces:
Attacks at first not mentioning names

Monteverdi persists!

(With Artusi claiming Monteverdi had his own notions of tuning approximating equal temperament, **many early music ensembles today** pick **quarter-comma mean tone tuning** for Monteverdi. So TET remains very ahead of the curve.)

Bach, Beethoven, and the Boys, David Barbers, Illustrations D. Donald, 1986

Monteverdi 1610 Vespers

Crucial transitional figure between the Renaissance and Baroque periods of music history

Monteverdi’s sin appeared to be his notions of tuning

https://global.oup.com/academic/product/the-monteverdi-vespers-of-1610-9780198164098?cc=us&lang=en& J Kurtzman 1999 Pages 488–494 Found at Vassar Music Library

Monteverdi’s antagonist, the theorist Giovanni Maria Artusi, claimed that Monteverdi had his own notions of tuning, which approximated equal temperament. Lindley cites a passage where Artusi complains that

> certain obstinate ‘modern composers’ (Monteverdi) entertained a theory of intonation according to which the C♯–B♭ is ‘neither a sixth nor a seventh, but sounds very well’ and F♯–B♭ ‘is a third’ and is divided into a Pythagorean whole-tone and two equal semitones’ as follows:[21]

https://en.wikipedia.org/wiki/Stile_antico refers to the Artusi Monteverdi controversy

Monteverdi 1610 Vespers

Crucial transitional figure between the Renaissance and Baroque periods of music history

Renaissance period: Very active discussion of Tuning Systems even back then

https://global.oup.com/academic/product/the-monteverdi-vespers-of-1610-9780198164098?cc=us&lang=en& J Kurtzman 1999 Pages 488–494 Found at Vassar Music Library

If the performance practice issues discussed in the preceding chapters seem ambiguous and incapable of definitive resolution, matters of tuning and temperament are equally problematic. Tempered tuning, mean-tone tunings, just intonation, Pythagorean tuning, and Ptolemaic tuning were all discussed and argued by theorists in the Renaissance and early seventeenth century.[1] Pythagorean tuning, with its pure fifths, seems to have been preferred until about the middle of the fifteenth century.[2] However, with increasing emphasis on the sonority of thirds and full triads in the late fifteenth century and the sixteenth century, mean-tone tuning, with its richer-sounding major thirds, became more popular, especially for organs and other keyboard instruments.[3] In the late sixteenth century Vincenzo Galilei, in discussing the intonation of singers, expressed his belief that 'the major third is formed by an irrational pro-

Bach Well-Tempered Klavier

1722, pictured 24 of 48 compositions

The conventional claim:
pieces made possible by the
huge breakthrough of
TET Equal Tuning system

Recent 2005 scholarship rebuts this claim:
that Bach used a tricky irregular
temperament and ducked TET !!

https://en.wikipedia.org/wiki/The_Well-Tempered_Clavier ***The Well-Tempered Clavier***, BWV 846–893, consists of two sets of preludes and fugues in all 24 major and minor keys for keyboard by Johann Sebastian Bach. I

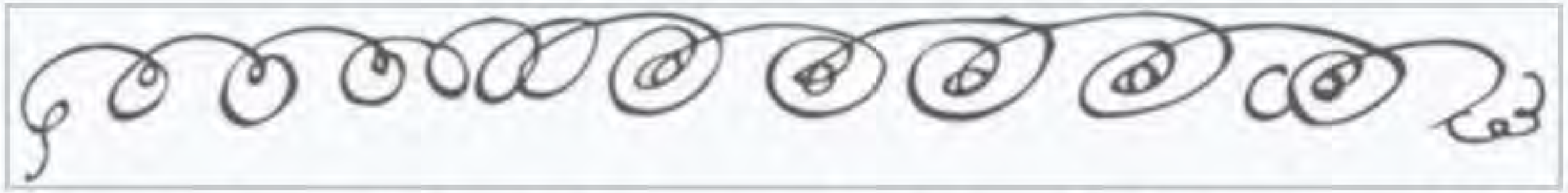
Bach Well-Tempered Klavier

1722

Very technical Wiki 2005

His tuning probably not so simple TET

“Rosetta Stone” 2005 paper – evidence that Bach may have used 1 /6 and 1/ 12 comma irregular temperament



Top of Bach's title page for the 1st book of *The Well-Tempered Clavier* (1722) showing handwritten loops which some have interpreted as tuning instructions.

https://en.wikipedia.org/wiki/The_Well-Tempered_Clavier

The Well-Tempered Clavier, BWV 846–893, consists of two sets of preludes and fugues in all 24 major and minor keys for keyboard by Johann Sebastian Bach. I

You’re Playing Bach Wrong

https://www.ethanhein.com/wp/2021/the-well-tempered-and-not-so-well-tempered-clavier/ 15Oct2021

https://www.youtube.com/watch?v=QEjANevZVfw You’re Playing Bach Wrong

Great explanation in 16 minutes of the Bach doodle question,
the 2005 Lehman “Bach’s Extraordinary Temperament: our Rosetta Stone” paper

Post 2005 revisions by music scholars:
Bach very likely did not use TET “even” temperament
for his “well”-tempered clavier pieces

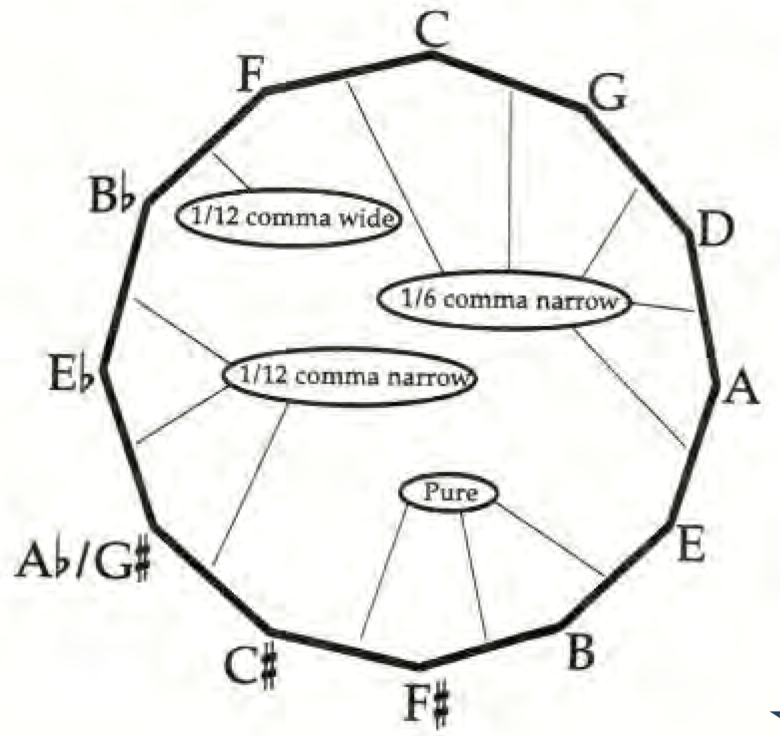
Irregular Temperament Fifth Circle

Lehman 2005 “Bach’s Extraordinary Temperament: our Rosetta Stone” Early Music 33 (2005.) Bach proposed to encode an irregular temperament that gives the closed circle

From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007 p37 For Bach see p148, 169 Rosetta Stone footnote.
Also see https://en.wikipedia.org/wiki/The_Well-Tempered_Clavier B. Lehman (2004, 2005)[22] proposed a

1 /6 and 1/ 12 comma layout derived from Bach's loops

Figure 7. Sample Irregular Temperament Fifth Circle.



“Rosetta Stone” Proposal 2005 that
Bach 48 well-tempered pieces
were actually customized, irregular temperaments
and not TET !

The Well-Tempered (but not-so-even-tempered) Clavier

https://www.ethanhein.com/wp/2021/the-well-tempered-and-not-so-well-tempered-clavier/ 15Oct2021

Bach wrote The Well-Tempered Clavier as a showcase for a new tuning system that could play in all twelve major and all twelve minor keys. Up until that point, the various European tuning systems only worked for some keys, not all of them. If you were in or near the key of C, you were usually okay, but as you moved further out on the circle of fifths, things got ugly fast. So this new tuning system that actually sounded good in all the keys was an exciting development.

However… no one knows what tuning system Bach used. **All we know is that it wasn't twelve-tone equal temperament, the one we all use now.** There were many systems in circulation at the time that people called "well temperament." Was Bach using Werckmeister? Kirnberger? Kellner? Some idiosyncratic system of his own invention? No one knows.

12-TET sounds fine throughout. But only fine. It's never offensive, but it lacks the spiciness and character of the other systems.

Quarter-comma meantone sounds good in C, but pretty terrible everywhere else. It has good-sounding thirds, but most of the fifths are flat, except for a few which are wildly sharp. So when meantone is in tune, it's very in tune, but when it's out, it's way, way out. The C major prelude mostly sounds nice in meantone, but C-sharp sounds horrendous. Fifth-comma sounds better than quarter, and sixth-comma sounds better than fifth.

Musical Acousticians building around Bach or Beethoven

https://archive.schillerinstitute.com/fidelio_archive/1992/fidv01n01-1992Wi/fidv01n01-1992Wi_047-the_foundations_of_scientific_mu.pdf Dec. 1992

A Brief History of Tuning

The first explicit reference to the tuning of middle C at 256 oscillations per second was probably made by a contemporary of J .S. Bach. It was at that time that **precise technical methods were developed**, making it possible to determine the exact pitch of a given note in cycles per second. **The first person said to have accomplished this was Joseph Sauveur** (1 653- 1 7 1 6), **called the father of musical acoustics**. He measured the pitches of organ pipes and vibrating strings, and defined the "ut" (nowadays known as "do") of the musical scale at 256 cycles per second.

J.S. Bach, as is well known, was an expert in organ construction and master of acoustics, and was in constant contact with instrument builders, scientists, and musicians all over Europe. So we can safely assume that he was familiar with Sauveur's work.

In Beethoven's time, the leading acoustician was Ernst Chladni (1 756- 1 827), whose textbook on the theory of music explicitly defined C = 256 as the scientific tuning. Up through the middle of the present century, C = 256 was widely recognized as the standard "scientific" or "physical" pitch.

In fact, A = 440 has never been the international standard pitch, and the first international conference …

Mozart's Father – A flat higher than G sharp ?!

From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007   p60

Leopold Mozart is mostly known today as the grouchy and overprotective father of one of the world's greatest musical geniuses. In Leopold's own day, his fame rested largely on his *Treatise on the Fundamental Principles of Violin Playing*, published in 1756, the year of Wolfgang's birth. Even today this treatise has caused him to be regarded, not just as Wolfgang's father but as the father of modern violin playing.

In discussing chromatic scales on the violin, Leopold reminds his readers that:

> . . . according to their proper ratios, notes with flat signs are a comma higher than those in the same position with a sharp sign. For example, D♭ is higher than C♯, A♭ higher than G♯, G♭ than F♯, and so on.
>
> Leopold Mozart, *Versuch einer gründlichen Violinschule* (1756)[3]

Semitone practice in the Mozart household

Extended Meantone Fifth Spiral – **Extra notes** A Flat G Sharp

From Ross Duffin, How Equal Temperament Ruined Harmony (and why you should care) 2007   p56   p53

Fifty-Five “Commas” in One Octave

5x9 = 45 commas plus 2x5 semitones

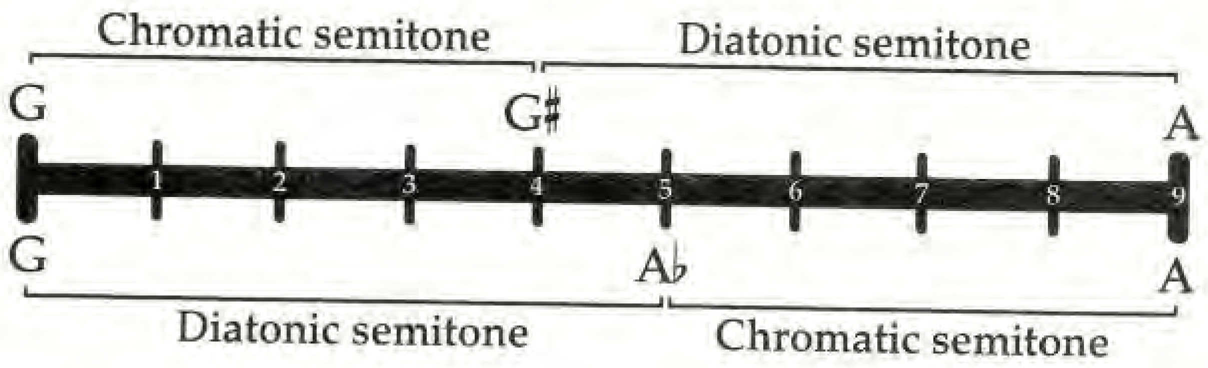
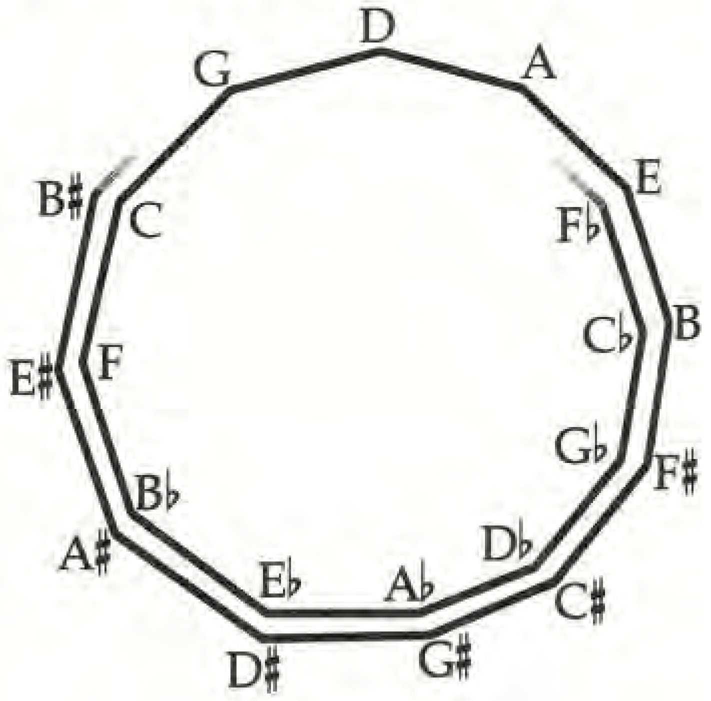


Figure 9. Sample Nine-Comma Whole Tone Within the 55-Division Octave.

Figure 10. Extended Meantone Fifth “Spiral.”



19th Century Physicist Helmholtz attack on Bach and Beethoven for abandoning natural tuning

https://archive.schillerinstitute.com/fidelio_archive/1992/fidv01n01-1992Wi/fidv01n01-1992Wi_047-the_foundations_of_scientific_mu.pdf

https://en.wikipedia.org/wiki/Helmholtz_pitch_notation

https://en.wikipedia.org/wiki/Sensations_of_Tone   Helmholtz 19th century

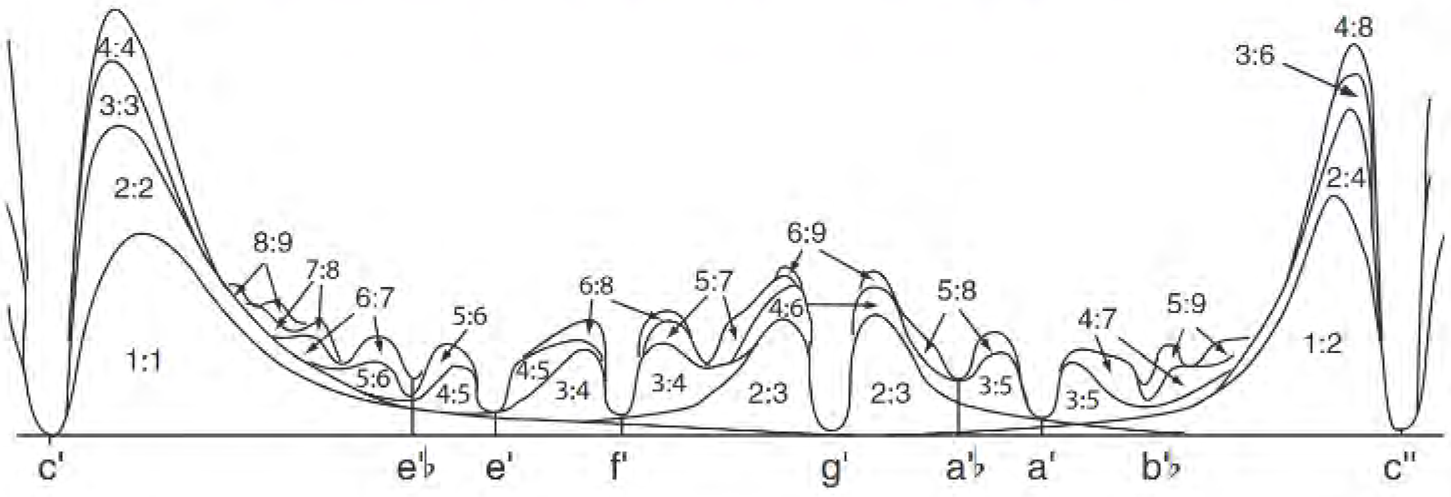
19th century physicist Helmholtz ….Arguing from this standpoint, Helmholtz demanded that musicians give up well-tempering and return to a " natural tuning" of whole-number ratios;

Helmholtz even attacked the music of J.S. Bach and Beethoven for being "unnatural" on account of their frequent modulations

Tuning and Timbre: A Perceptual Synthesis Bill Sethares

https://sethares.engr.wisc.edu/paperspdf/HelmTTSS.pdf no date

Helmholtz's Dissonance Curve



Two pitches are sounded simultaneously. The regions of roughness due to pairs of interacting partials are plotted over one another, leaving only a few narrow valleys of relative consonance. The figure is redrawn from *On the Sensation of Tone*.

Tuning and Timbre:
A Perceptual Synthesis
Bill Sethares

https://sethares.engr.wisc.edu/paperspdf/HelmTTSS.pdf no date

Adaptive Tuning
retains simple ratios
while avoiding wandering pitch

(thought to need a special digital keyboard)

JI vs. 12-tet vs. Adaptive Tuning

An example of drift in Just Intonation: the fragment ends about 21 cents lower than it begins. 12-tet maintains the pitch by distorting the simple integer ratios. The adaptive tuning microtonally adjusts the pitches of the notes to maintain simple ratios and to avoid the wandering pitch. Frequency values are rounded to the nearest 0.5 Hz. (sytonJIdrift, synton12tet, syntonadapt)

Frequencies when played in JI with held notes:	392.5	436---	436	387.5--	387.5
	327	327	290.5-	290.5	323
	261.5-	261.5	290.5	242	258.5
	131	109	87	96.5	129
Frequencies when played in 12-tet:	392	440	440	392	392
	329.5	329.5	293.5	293.5	329.5
	261.5	261.5	293.5	247	261.5
	131	110	87.5	98	131
Frequencies when played in adaptive tuning:	392.5	440	438.5	391	392.5
	327	330	292	294	327
	261.5	264	292	245	261.5
	131	110	87.5	98	131
Ratios when played in adaptive tuning and in JI:	6/5	4/3	3/2	4/3	6/5
	5/4	5/4	1/1	6/5	5/4
	2/1	6/5	5/3	5/4	2/1

39

How Equal Temperament Ruined Harmony Ross Duffin 2006

https://www.kirkusreviews.com/book-reviews/ross-w-duffin/how-equal-temperament-ruined-harmony/

https://books.google.com/books/about/How_Equal_Temperament_Ruined_Harmony_and.html

How Equal Temperament Ruined Harmony (and why You Should Care)

Ross W. Duffin

W. W. Norton & Company, 2007 - Music - 196 pages

Ross W. Duffin presents an engaging and elegantly reasoned exposé of musical temperament and its impact on the way in which we experience music. A historical narrative, a music theory lesson, and, above all, an impassioned letter to musicians and listeners everywhere, *How Equal Temperament Ruined Harmony* possesses the power to redefine the very nature of our interactions with music today.

For nearly a century, equal temperament--the practice of dividing an octave into twelve equally proportioned half-steps--has held a virtual monopoly on the way in which instruments are tuned and played. In his new book, Duffin explains how we came to rely exclusively on equal temperament by charting the fascinating evolution of tuning through the ages. Along the way, he challenges the widely held belief that equal temperament is a perfect, "naturally selected" musical system, and proposes a radical reevaluation of how we play and hear music.

Chinese Musicology, pentatonic scale

https://en.wikipedia.org/wiki/Chinese_musicology samples of scales given Yu Shang Gong Jue Zhi

The ancient Chinese defined, by mathematical means, a gamut or series of 十二律 (Shí-èr-lǜ), meaning twelve lǜ, from which various sets of five or seven frequencies were selected to make the sort of "do re mi" major scale familiar to those who have been formed with the Western Standard notation. The 12 *lü* approximate the frequencies known in the West as the chromatic scale, from A, then B-flat, through to G and A-flat.

The first musical scales were derived from the harmonic series. On the Guqin (a traditional instrument) all of the dotted positions are equal string length divisions related to the open string like 1/2, 1/3, 2/3, 1/4, 3/4, etc. and are quite easy to recognize on this instrument. The Guqin has a scale of 13 positions all representing a natural harmonic position related to the open string.

Scale and tonality

Most Chinese music uses a pentatonic scale, with the intervals (in terms of *lǜ*) almost the same as those of the major pentatonic scale. The notes of this scale are called *gōng* 宫, *shāng* 商, *jué* 角, *zhǐ* 徵 and *yǔ* 羽. By starting from a different point of this sequence, a scale (named after its starting note) with a different interval sequence is created, similar to the construction of modes in modern Western music.

Since the Chinese system is not an equal tempered tuning, playing a melody starting from the *lǜ* nearest to A will not necessarily sound the same as playing the same melody starting from some other *lǜ*, since the wolf interval will occupy a different point in the scale. The effect of changing the starting point of a song can be rather like the effect of shifting from a major to a minor key in Western music. The scalar tunings of Pythagoras, based on 2:3 ratios (8:9, 16:27, 64:81, etc.), are a western near-parallel to the earlier calculations used to derive Chinese scales.

Pentatonic scale

https://en.wikipedia.org/wiki/Pentatonic_scale

A **pentatonic scale** is a musical scale with five notes per octave, in contrast to heptatonic scales, which have seven notes per octave (such as the major scale and minor scale).
Pentatonic scales were developed independently by many ancient civilizations[2] and are still used in various musical styles to this day. As Leonard Bernstein put it: "the universality of this scale is so well known that I'm sure you could give me examples of it, from all corners of the earth, as from Scotland, or from China, or from Africa, and from American Indian cultures, from East Indian cultures, from Central and South America, Australia, Finland ...now, that is a true musico-linguistic universal."[3] There are two types of pentatonic scales: those with semitones (hemitonic) and those without (anhemitonic).

Arab tone system uses 24, not 12, divisions!

https://en.wikipedia.org/wiki/Arab_tone_system

Samples given for Arab tone system

Arab tone system

From Wikipedia, the free encyclopedia

The modern **Arab tone system**, or system of musical tuning, is based upon the theoretical division of the octave into twenty-four equal divisions or 24-tone equal temperament (24-TET), the distance between each successive note being a quarter tone (50 cents). Each tone has its own name not repeated in different octaves, unlike systems featuring octave equivalency. The lowest tone is named *yakah* and is determined by the lowest pitch in the range of the singer. The next higher octave is *nawa* and the second *tuti*.[1] However, from these twenty-four tones, seven are selected to produce a scale and thus the interval of a quarter tone is never used and the three-quarter tone or neutral second should be considered the characteristic interval.[2]

Stringed Instruments – Piano, Harp

Stringed instruments have very clever designs to make them compact!

Frequency f in beats per second
Piano has 7 **octaves** powers of two
Lowest C   65.406 beats/sec
Middle C = 65.406   x 2 x 2 x 2 or 523.25 beats/s
Highest C   65.406   x 2 x 2 x 2 **x 2 x 2 x 2 x 2**
8372.02 = 65.406 x 128

Most Obvious Factor – length of the string
Tricky: Tension of strings, mass of strings
Piano Middle C = 630mm our piano
Piano Highest C = 76mm our piano
630/76 = **8.3X**   8372.02 beats / 523.25beats = **16X**
The Piano people are being clever!   Getting a 16X frequency range while only lengthening the strings by 8.3X
In more detail, Middle C 1.9X length, next 1.5X, next 1.6X, next 1.7X where the frequency ratio is 2.0X per octave

If the length of the string is L, the fundamental harmonic is the one produced by the vibration whose nodes are the two ends of the string, so L is half of the wavelength of the fundamental harmonic. Hence one obtains Mersenne's laws:

$$f = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

where T is the tension (in Newtons), μ is the linear density (that is, the mass per unit length), and L is the length of the vibrating part of the string. Therefore:

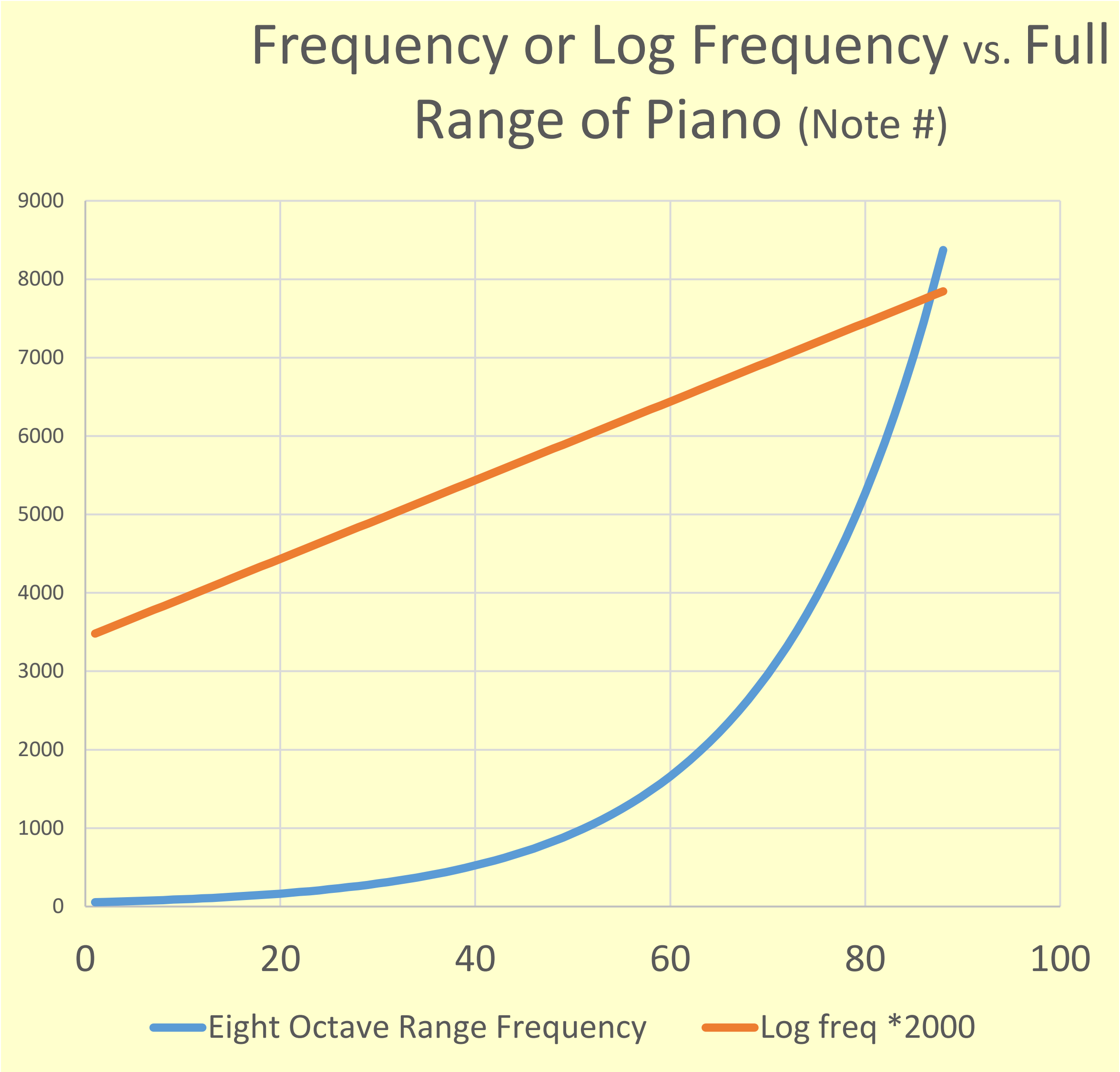
- the shorter the string, the higher the frequency of the fundamental
- the higher the tension, the higher the frequency of the fundamental
- the lighter the string, the higher the frequency of the fundamental

Frequency versus note number

for the 88 notes of piano

Musical Scale is logarithmic and so has a huge range!

Lowest note, A is 55 hz
A near middle is A440 = 55x8 = 55 x 2^3
Middle C is 523.25
Highest A is 7040hz = 55x128 = 55 x 2^7
Highest note, C is 8372hz



Lyon & Healy Harp – string lengths revisited 2020

Lyon and Healy Harp string lengths measured daughter's lever harp in 2005, thinking of physics of music

Replotting the 2005 measurements 2019, in reference to ideas for science Olympiads 36 = 5x7+1 strings starting with C-65.4hz

Even tone scale -- notes increase by the 12th root of two, an octave doubles frequency, A440 vibrations/sec is just below middle C, piano has notes #1-88

Cell entry formula: the note A-sharp is E11 = E10*2^(1/12) or 55.000 * 1.059463... = 58.27047...

G is a **fifth** or 7 half steps or 1.49831 or **3/2** times above C. F a **fourth** or 5 half steps or 1.33484 or about **4/3** above C. E a **third** or 1.2599 or about **5/4** above C

Note		Eight Octave Range Frequency	Harp String Length, mm	1/L *500
A	1	55.0000000000		
D-#	7	77.7817459305	49.313	10.13942
G-#	12	103.8261743950	43.125	11.5942
A-#	14	116.5409403795	42.125	11.86944
D-#	19	155.5634918610	34.75	14.38849
G-#	24	207.6523487900	27.125	18.43318
A-#	26	233.0818807590	25.125	19.9005
D-#	31	311.1269837221	19.5	25.64103
G-#	36	415.3046975799	15.125	33.05785
A	37	440.0000000000		
A-#	38	466.1637615181	14	35.71429
D-#	43	622.2539674442	10.875	45.97701

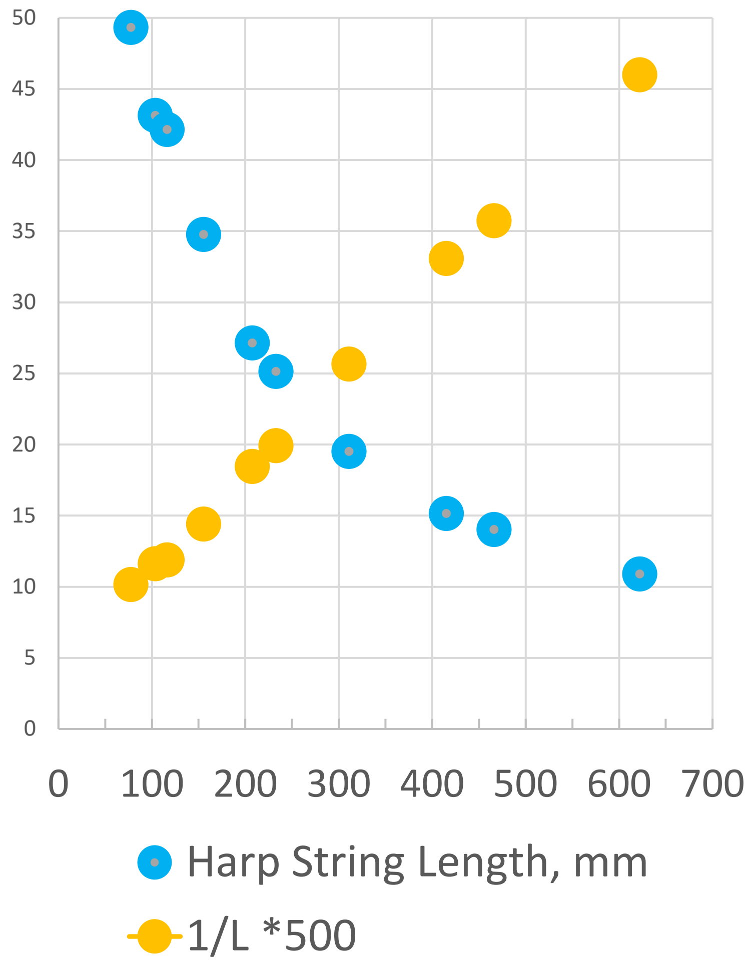
Instruments seem to have simple inverse length relationship with frequency, but are actually quite tricky in minimizing instrument size!

Less than full 2X length difference per octave! 1.4X or 1.8X for harp

1.9X or 1.5X or 1.7X for upper range piano

https://en.wikipedia.org/wiki/String_vibration

Harp String Length vs. Frequency



Baby Grand Piano string lengths

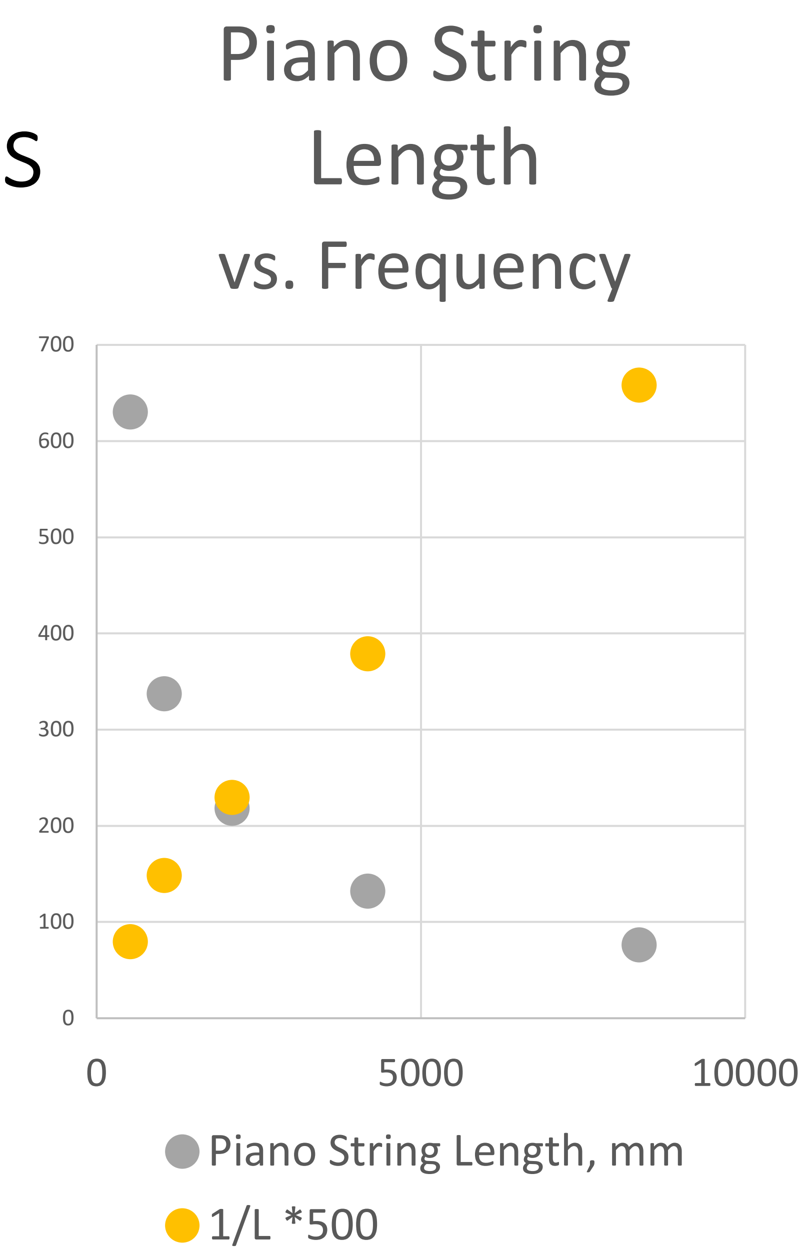
Getting out with tape measure on family piano and harp

Note		Eight Octave Range Frequency	Piano	Piano String Length, mm	1 / L *50000
A	1	55.0000000000	A-zero		
C	40	523.2511306012	C-4 Middle C	630.0	79.4
C	52	1046.5022612024	C-5	337.0	148.4
C	64	2093.0045224048	C-6	218.0	229.4
C	76	4186.0090448096	C-7	132.0	378.8
C	88	8372.0180896192	C-8	76.0	657.9

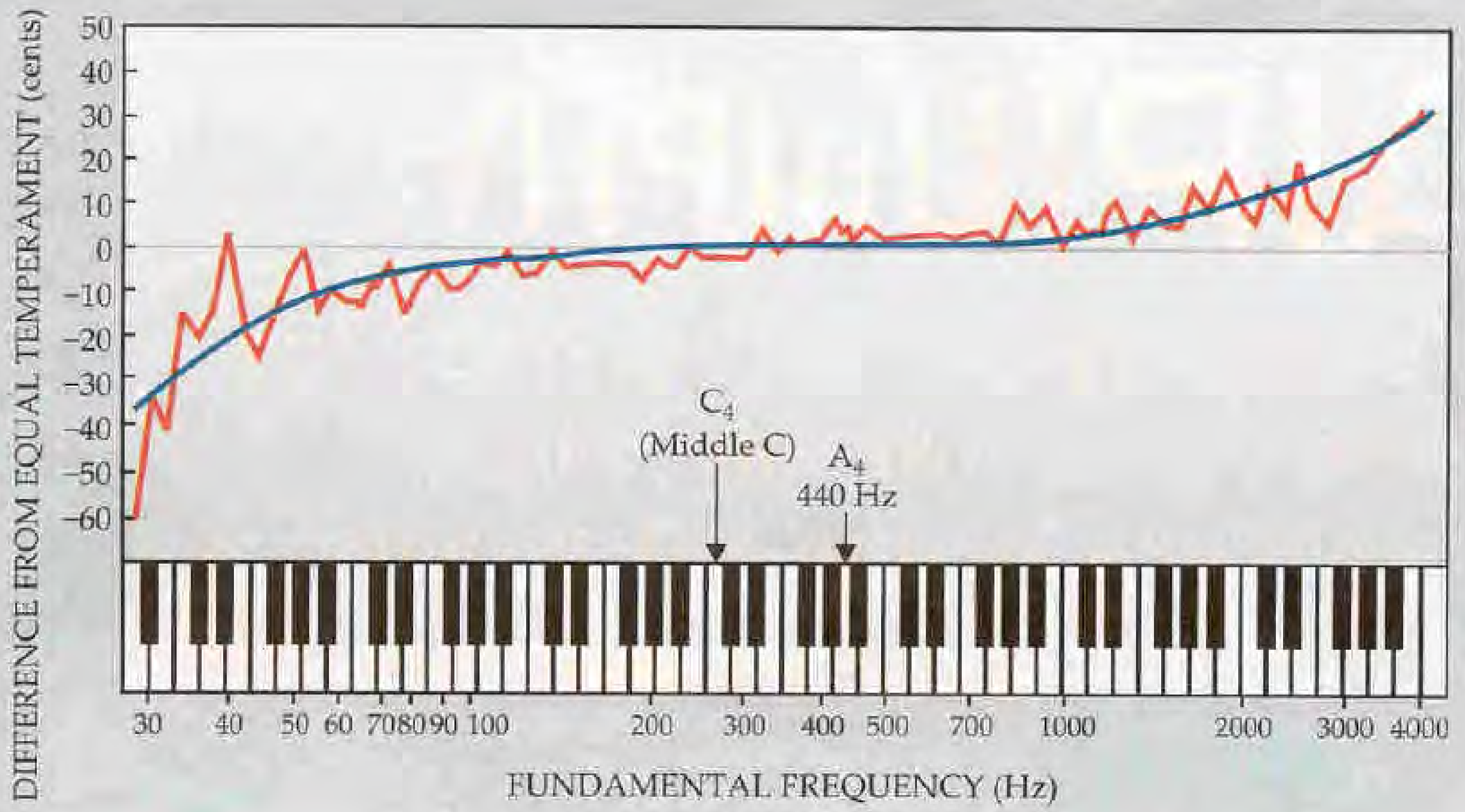
Instruments seem to have simple inverse length relationship with frequency, but are actually quite tricky in minimizing instrument size!

Less than full 2X length difference per octave! 1.4X or 1.8X for harp

1.9X or 1.5X or 1.7X for upper range piano



Piano Tuning – R. Feynman writes unique critique, Physics Today John Bryner Dec. 2009 p 46



Pianos sound better when they are tuned by ear slightly away from equal temperament, as shown in a 1961 study by Daniel Martin and W. Dixon Ward. The red curve shows the aural tuning of the technician used in the Martin and Ward work. The blue curve is a comparative standard that averages over many aurally tuned pianos. In equal temperament, the frequencies of two neighboring notes on the piano differ by 100 cents. (Adapted from D. W. Martin, W. D. Ward, *J. Acoust. Soc. Am.* **33**, 582, 1961.)

Piano Tuning – R. Feynman writes unique critique, Physics Today
John Bryner Dec. 2009 p 46

July 3, 1961

Dear Mr. McQuigg

I figured out the effect of wire stiffness on the vibration frequency of strings. The mathematical formula is [note 1]

$$\text{true frequency} = f\left(1 + \frac{\pi}{2}\,\frac{EA^2\mu}{T^2}\,f^2\right),$$

where f is the frequency you would get forgetting about stiffness (for fundamental

$$f = \frac{2}{l}\sqrt{\frac{T}{\mu}}$$

[note 2] for string of length l), T is the tension in the string, A is the area of the string cross-section, E is Young's modulus of steel (measures the stiffness of the wire), μ is the weight of wire per unit length = 7.80 grams × A in squ. centimeters for steel. I have worked this out roughly for steel wires—*not* for the weighted bass strings. It says that the frequency is shifted by

$$\frac{1}{20}\ \text{cent} \cdot \frac{(\text{diameter of string in mm})^6}{(\text{tension}/150\ \text{lbs})^2} \cdot \left(\frac{\text{frequency}}{100\ \text{Hz}}\right)^2$$

That is for middle C_4, say, supposing T = 150 lbs and the

How to tune a piano

Musicians have developed their own jargon for naming musical notes and relations between them. Before reviewing the basics of piano tuning, I'll define some of that jargon with the help of the treble portion of the piano keyboard illustrated below.

Any two neighboring notes on the piano are said to differ by a semitone interval. Twelve semitones span an octave, and you can see in the illustration that the pattern of piano keys repeats after each octave. Two notes that differ by a number of octaves are given the same letter name; the octave is distinguished by a subscript. So, the interval separating C_4 and C_5 is an octave. Richard Feynman's letter discusses at length the relation between C_4 and G_7, which are identified on the keyboard. The illustration also specifies F_4 (discussed later) and A_4, the note sounded by the oboe to tune up an orchestra.

In all systems of tuning, every pitch may be derived from its relationship to a standard. In the case of piano tuning, the usual choice is to assign the frequency 440 Hz to the note A_4. The frequencies of all the other notes are set by counting the beat rates that originate in upper, nearly coincident overtones when two notes are struck simultaneously. One might have thought that a piano could be tuned with the frequencies of any two notes related by simple whole-number ratios. Then all pairs of notes would be separated by "pure intervals," and one wouldn't need to worry about beat counting. It is mathematically impossible, however, to have only pure intervals in a standard 13-note-per-octave keyboard. Some of the intervals must be altered, which results in beating. Those altered tunings are referred to as temperaments.

Equal temperament is a system of tuning keyboard instruments in which the frequency ratio for any two notes separated by a semitone is $2^{1/12}$. Rather different, unequal temperaments are sometimes used for historical reasons. All temperaments are modified in piano tuning because the steel strings have nonzero stiffness, which causes the overtones to be higher in frequency than for simple harmonics; the effect is called inharmonicity. To quantify small frequency changes, piano tuners divide the semitone interval into 100 cents. The frequency ratio of two notes that differ by c cents is thus $2^{c/1200}$. As a result of inharmonicity, all keyboard intervals are stretched in frequency. Variations from equal temperament are almost negligible in the middle of the keyboard, but for a small, aurally tuned piano they rise to about 30 cents sharp at the treble end and about 30 cents flat at the bass. Larger pianos generally have less stretch, but it is always present.

Most piano tuners nowadays, including me, regularly use an electronic tuning device. The ETD makes the tuning process simpler and less demanding on the ears. It usually produces good results, but we sometimes need to make corrections to render the tuning aurally acceptable. On the other hand, an ETD can detect minor flaws in an aural tuning. There are piano technicians with strong preferences on each side of the aural versus ETD divide. Aural tuning uses the ear—the ultimate judge of what sounds good. But human fatigue can make it difficult to tune in a noisy environment or to duplicate results. The ETD works well in noisy environments, never gets tired, and makes duplication easy. Generally, one method is strong where the other is weak and many tuners prefer to use the best of both.

Aurally tuning a piano consists of three steps. The first is to establish the proper pitch for one note, usually A_4. Next is to tune the temperament octave, usually the octave between F_3 and F_4 (F_3, an octave below F_4, is not in the portion of the piano keyboard illustrated here). At last, using the temperament octave as a standard, one can tune the rest of the piano. Note that when a piano key is pressed, two or three strings inside the piano are struck. A piano technician needs to tune each string individually.

A simplified version of a tuning procedure goes something like this. Two of the three strings of each note in the temperament octave are muted so that only one will vibrate when the corresponding piano key is played. Then a 440-Hz tuning fork is sounded and the tension in the A_4 string is adjusted so that no beats are heard. Next, one tunes A_3 by playing A_3 and A_4 together and adjusting the tension in the A_3 string until no beats are heard. Because of inharmonicity, the octave interval A_3–A_4 will be slightly wider than 1200 cents.

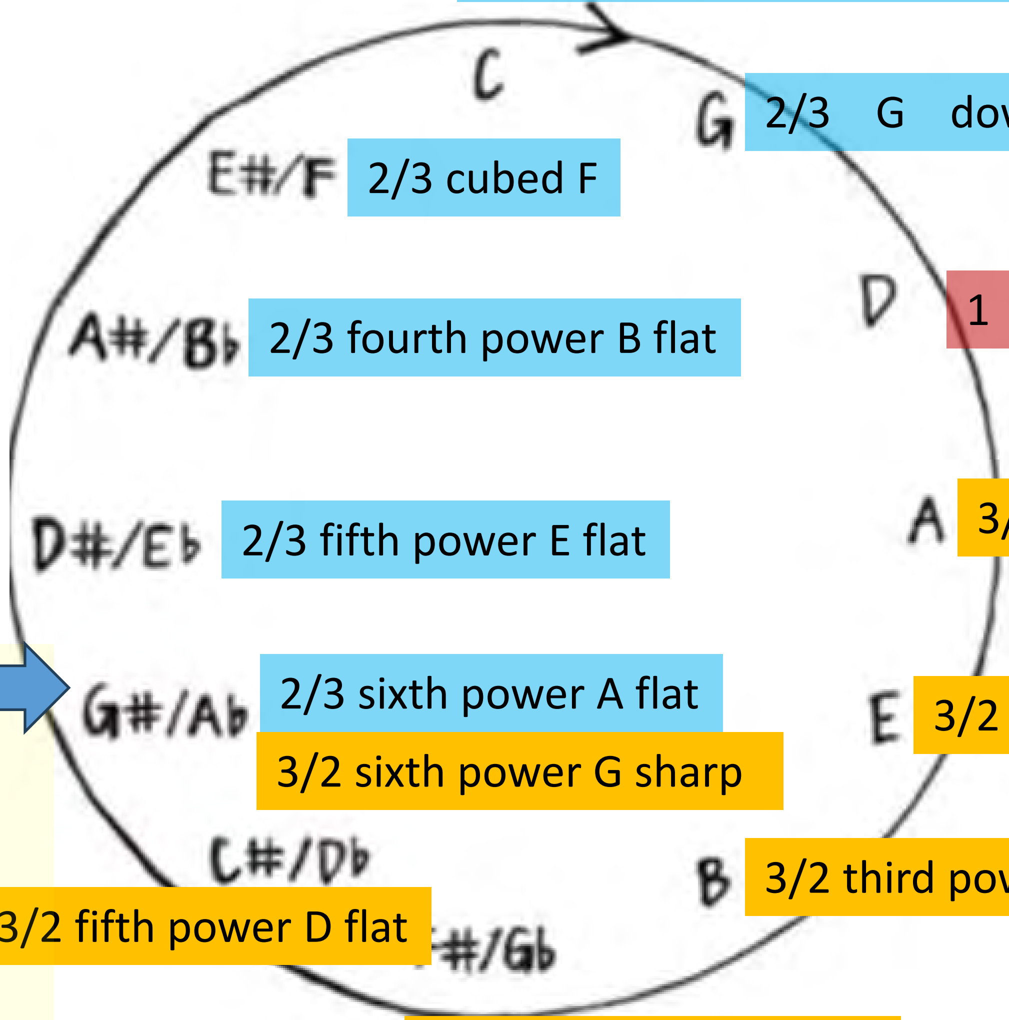
Next, the "major third" interval F_3–A_3 is tuned. In equal temperament, the frequency ratio of notes separated by a major third is $2^{4/12}$ (1.2599), about 14 cents wider than the pure major third interval of 5/4. The result is that a piano tuner who plays F_3 and A_3 together will hear about 7 beats per second. The beat rate for major thirds increases as one goes up the keyboard. It doubles to 14 beats per second for the next octave F_4–A_4, doubles again for F_5–A_5, and so forth. The goal in setting the temperament octave is not to count theoretical beat rates exactly but to make them progress evenly through the octave. Tuners use various means to accomplish that goal.

After the temperament octave has been set and tested, it serves as the standard for tuning the remaining notes of the piano. One simply plays octaves and adjusts the tension in the untuned note until there are no beats. If the process is carried out carefully, the piano will sound good. In the end, tuning a piano is an art as well as a science.

This page – the exciting moment when Pythagoras suddenly became simple genius, if viewed correctly within Circle of Fifths

Circle of Fifths

Powers of 3/2 or 2/3 Pythagoras



Our Famous "Comma"

A flat "Diminished fifth"

412.03 setting A440

G sharp "Augmented Fourth"

417.65 setting A440

SWEDEN "Just" organ - two keys

TET keyboards – skip 2nd key

Musical Instruments and Twelve Tone Equal Temperament - Conclusions

Why was I intrigued as child in this esoteric subject? Why should anyone else be?

- As a child taking piano lessons: fascinated that "TET" equal temperament forces all keys "equivalent" C vs. C sharp vs. F sharp
  - **But:** musical notation mysteriously super-redundant
  - **But:** Composers passionate about particular keys: Beethoven Quartet C sharp minor
  - **But:** Musicians hear resonances & overtones
- **What We've Learned?**
  - **Pythagoras fantastic insight 6th century BC – Octave, Circle of Fifths 3/2 Ratio, Base 12**
    - **Centered on chord richness, resonances, and overtones**
  - **TET – Octave Frequency Double 2.000 - all other intervals slightly out of tune**
    - **Music Re-purposed: Centered on Modulations, Composer's Needs, Instrument Standardization**
    - **Bach, Mozart, Beethoven score spectacular success!**
    - **Physicist Helmholtz leads a groundswell of discontent with TET's major third loss of resonances & overtones**
  - **Beyond TET**: About cellist Pablo Casals p155 "it's so beautiful but why does he play out of tune?" Casals taught "expressive intonation." Guarneri Quartet p74 reaches beyond a "sterile and static" equal temperament. Good unaccompanied choirs tweak for overtones. ***Great musicians recapture rich chords & sounds by reaching beyond TET***

NET: Musicianship is subtle & evolving, beyond the genius platforms of Pythagoras and TET.

My personal inspiration for creating this presentation was hearing the meantone Sweden 1651 *Düben* Organ